CLASSIFICATION OF EMBEDDINGS BELOW THE METASTABLE DIMENSION

A. Skopenkov

Abstract. It is developed a new approach to the classical problem on isotopy classification of embeddings of manifolds into Euclidean spaces. This approach involves studying of a *new embedding invariant*, of *almost-embeddings* and of *smoothing*, as well as *explicit constructions* of embeddings. Using this approach we obtain complete concrete classification results *below the metastable dimension range*, i.e. where the configuration spaces invariant of Haefliger-Wu is incomplete. Note that all known complete concrete classification results, except for the Haefliger classification of links and smooth knots, can be obtained using the Haefliger-Wu invariant.

More precisely, we classify embeddings $S^p\times S^{2l-1}\to\mathbb R^{3l+p}$ for $p<l$ in terms of homotopy groups of Stiefel manifolds (up to minor indeterminancies for $p>1$ and for the smooth category). A particular case states that *the set of piecewise-linear isotopy classes of piecewise-linear embeddings (or, equivalently, the set of almost smooth isotopy classes of smooth embeddings) $S^1\times S^3\to\mathbb R^7$ has a geometrically defined group structure, and with this group structure is isomorphic to $\mathbb Z\oplus\mathbb Z\oplus\mathbb Z_2$.*

We exhibit *an example disproving the conjecture proposed by Viro and others* on the completeness of the multiple Haefliger-Wu invariant for classification of PL embeddings of connected manifolds in codimension at least 3.

1. Main results

Motivation.

This paper is on the classical Knotting Problem: *for an n-manifold N and a number m describe isotopy classes of embeddings $N\to\mathbb R^m$*. For recent surveys [RS99, Sk07]; whenever possible we refer to these surveys not to original papers.

If N is closed d-connected, then the Knotting Problem is easier for $2m\ge 3n+4$ in the smooth category and for

$$2m\ge 3n+3-d$$

in the piecewise-linear (PL) category. This is so because the *Haefliger-Wu invariant* is bijective (by van Kampen, Shapiro, Wu, Haefliger, Weber and the author [RS99, §4, Sk07, §5]). This simplest configuration space invariant is defined below in §1. If N is closed d-connected, then the Knotting Problem is much harder for

$$2m\le 3n+2-d$$

both in PL and smooth categories. Indeed, for N distinct from disjoint union of (homology) spheres no concrete complete description of isotopy classes was known (except recent results

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[KS05, Sk06, CRS]), in spite of the existence of interesting approaches [Br68, Wa70, GW99, We]. In this paper the case $2m = 3n+2-d$ is studied, i.e. the first case when $2m \le 3n+2-d$.

Many interesting counterexamples in the theory of embeddings [Al24, Ko62, Hu63, Wa65, Ti69, BH70, Bo71, MR71, Sk02, CRS04] (see also the Whitehead Torus Example 1.7 below) are embeddings $S^p \times S^q \to \mathbb{R}^m$, i.e. *knotted tori.* Classification of knotted tori is a natural next step (after the link theory [Ha66']) towards classification of embeddings of *arbitrary* manifolds. Thus classification of knotted tori gives some insight or even precise information concerning arbitrary manifolds (cf. [Hu63, Hu69, §12, Vr77, Sk05]) and reveals new interesting relations to algebraic topology. Since the general Knotting Problem is recognized to be unsolvable, it is very interesting to solve it for the important particular case of knotted tori.

Informal description of main results.

Our main results are the Main Theorems 1.3 and 1.4 below, which are *classification of embeddings* $S^p \times S^{2l-1} \to \mathbb{R}^{3l+p}$ for $0 < p < l$ in terms of homotopy groups of Stiefel manifolds (for $p > 1$ or for the smooth case up to minor indeterminacy). An interesting particular case is Theorem 1.3.aPL for $k = 1$: this is the first complete concrete classification of embeddings of a *non-simply-connected* 4-manifold into $\mathbb{R}^7$, cf. [KS05, Sk05]. Our ideas can be extended to give rational classification for $2m < 3n + 2 - d$ [CRS] or estimations for arbitrary manifolds [Sk05].

A nice feature of our classification results is an explicit and simple construction of representatives for all isotopy classes, see §2. The simplicity of these constructions suggests that such embeddings could well appear in other branches of mathematics. These constructions are used in the proof and imply interesting results such as Symmetry Remark in the appendix.

The main difficulty in obtaining such classification is that the Haefliger-Wu invariant is not complete in our situation. This is shown by the Whitehead Torus Example 1.7 *disproving the conjecture proposed by Viro* on the completeness of the multiple Haefliger-Wu invariant for classification of PL embeddings of connected manifolds in codimension at least 3 (precise formulation is given below; the same or analogous conjectures were proposed by Dranishnikov, Szücs, Schepin and perhaps others). The construction (but not the proof) of this example is very simple and explicit, see the end of §1.

The main results are obtained by studying a *new embedding invariant*, *almost-embeddings*, *smoothing* and *explicit construction* of all the isotopy classes of embeddings. The construction of the new invariant is based on known ideas (for references see §2). Thus the main point of this paper is not the invention of a new invariant but *proof of new non-trivial properties of the invariant, allowing to obtain concrete complete classification results.* These ideas presented in §2 are hopefully interesting in themselves.

A difficult point is that at some points we use smoothing theory to deduce a result in the PL category from a result in *almost smooth* category, and at other points vice versa. The results in the smooth category are obtained using those in almost smooth category and then applying and developing smoothing theory, see §8.

Plan of the paper and notation.

In the rest of §1 we present statements of main results. Although they are independent on the previous work, we review most closely related known results in §1. The two remaining subsections of §1 could be read independently on each other, although in fact they are interrelated. Remarks of §1 (and of other sections) can be omitted during the first reading.

In §2 the main ideas are exposed (except for smoothing) and used to deduce the main results in the PL and almost smooth categories. The theorems used for this deduction are

proved in §§3–7. In §8 smoothing theory is developed and applied to prove the main results in the smooth category.

Sections §3–§8 are independent on each other in the sense that if in one section we use the result proved in the other, then we do not use the proof but only the statement which is presented in §2 (except that in §6 we use part of the Restriction Lemma 5.2). So a reader can pick up one of the main results of §1 (or a specific case in a specific category), read its proof in §2 and then read the proof of only those parts of §§3–8 which are required.

Let us fix some notation. Fix a codimension 0 ball B in a smooth manifold. A piecewise smooth embedding of this manifold is called *almost smooth* if it is a smooth embedding outside this ball. A piecewise smooth concordance of this manifold is called *almost smooth* if it is a smooth embedding outside $B \times I$.

Denote CAT = DIFF (smooth) or PL (piecewise linear) or AD (almost smooth). (The study of AD embeddings is not only interesting in itself, but is required to prove the main results in PL and DIFF categories.) If CAT is omitted, then a statement holds in all the three categories, unless specified otherwise. Let $\mathrm{Emb}^m_{CAT}(N)$ be the set of CAT embeddings $N \to \mathbb{R}^m$ up to CAT isotopy. For terminology concerning PL topology we refer to [RS72].

Denote by $V_{k,l}$ the Stiefel manifold of l-frames in $\mathbb{R}^k$. Denote by $\mathbb{Z}_{(s)}$ the group $\mathbb{Z}$ for s even and $\mathbb{Z}_2$ for s odd. This notation should not be confused with the notation for localization. Denote $\pi^S_i = 0$ for $i < 0$.

Knotted tori.

Denote

$$C^{m-q}_q := \mathrm{Emb}^{\,m}_{DIFF}(S^q) \qquad \text{and} \qquad KT^m_{p,q,CAT} := \mathrm{Emb}^{\,m}_{CAT}(S^p \times S^q).$$

The 'connected sum' commutative group structure on C^{m-q}_q and on $KT^m_{0,q}$ was defined for $m \geq q+3$ in [Ha66, Ha66']. In §2 we define an 'S^p-parametric connected sum' commutative group structure on $KT^m_{p,q,CAT}$ for $m \geq 2p+q+3$ or $(m,p,q) = (7,1,3)$, cf. [Sk05].

Concrete complete classification of knotted tori was known only

$$\text{either if } p = 0 \text{ and } 3m \geq 4q+6, \quad \text{or if } 2m \geq 3q+2p+4 \text{ and } p \leq q.$$

We state it as the Haefliger Theorem 1.1 and Theorem 1.2. Our main results are the Main Theorems 1.3 and 1.4 that give *classification of knotted tori for*

$$2m = 3q+2p+3 \quad \text{and} \quad 1 \leq p < q/2.$$

The Haefliger Theorem 1.1.

(PL) The group of PL embeddings $S^q \sqcup S^q \to \mathbb{R}^m$ up to PL isotopy is

$$KT^m_{0,q,PL} \cong \pi_q(S^{m-q-1}) \oplus \pi_{2q+2-m}(V_{M+m-q-1,M}) \quad \textit{for} \quad 3m \geq 4q+6,$$

where M is large enough [Sk07, Theorem 3.6.b, Sk07'].

(DIFF) [Sk07, Theorem 3.6.a] The group of smooth embeddings $S^q \sqcup S^q \to \mathbb{R}^m$ up to smooth isotopy is

$$KT^m_{0,q,DIFF} \cong KT^m_{0,q,PL} \oplus C^{m-q}_q \oplus C^{m-q}_q \quad \textit{for} \quad m-q \geq 3.$$

Remarks. (a) There is a description of $\mathrm{Emb}^m(S^{p_1}\sqcup\dots\sqcup S^{p_s})$ for $m-3\ge p_1,\dots,p_s$ in terms of exact sequences [Ha66', Ha86]. For $3m<4q+6$ this description is not as explicit as for the case $3m\ge 4q+6$ above, but it seems for the author to be 'the best possible': it reduces the classification of links to certain calculations in homotopy groups of spheres, and when such calculations cannot be completed, then no other approach would give an explicit answer. However, an alternative classification of links (e.g. using β-invarant of this paper [Sk07'] or isovariant configuration spaces [Me]) could give clearer construction of the invariants and could be applicable to related problems [Sk07', Me].

(b) Particular cases of the Haefliger Theorem 1.1PL state that $KT^m_{0,q,PL}\cong\pi^S_{2q+1-m}$ for $2m\ge 3q+4$ [Ze62, Ha62'] and that $KT^{3l}_{0,2l-1,PL}\cong\pi_{2l-1}(S^l)\oplus\mathbb{Z}_{(l)}$ for $l\ge 2$ [Ha62', Ha66', Theorem 10.7].

Theorem 1.2. *(PL) [Sk07, Theorem 3.9] The set of PL embeddings $S^p\times S^q\to\mathbb{R}^m$ up to PL isotopy is*

$$KT^m_{p,q,PL}=\pi_q(V_{m-q,p+1})\oplus\pi_p(V_{m-p,q+1})\quad\textit{for}\quad p\le q\quad\textit{and}\quad m\ge 3q/2+p+2.$$

(DIFF) The group of smooth embeddings $S^p\times S^q\to\mathbb{R}^m$ up to smooth isotopy is

$$KT^m_{p,q,DIFF}\cong\pi_q(V_{m-q,p+1})\oplus C^{m-p-q}_{p+q}\quad\textit{for}\quad p\le q\quad\textit{and}\quad m\ge\max\{2p+q+3,3q/2+p+2\}.$$

Remarks. (a) Theorem 1.2DIFF was only known for $2m\ge 3q+3p+4$, when $C^{m-p-q}_{p+q}=0$ and so $KT^m_{p,q}\cong\pi_q(V_{m-q,p+1})$ [Sk02, Corollary 1.5] (or for $p=0$). The extension to $2m<3q+3p+4$ is a new result of this paper (proved in §8) but not the main one. See [KS05, Sk06] for motivation (and explanation why Theorem 1.2DIFF is not a trivial corollary of Theorem 1.2PL and smoothing theory). Cf. [Sk06, Higher-dimensional Classsification Theorem (a)].

(b) Theorem 1.2 is a generalization to $p>0$ of the case $2m\ge 3q+4$ of the Haefliger Theorem 1.1.

(c) The restriction $m\ge 2p+q+3$ is sharp in Theporem 1.2DIFF (because the analogue of Theorem 1.2DIFF for $m=2p+q+2$ is false by [Sk06, Classification Theorem and Higher-dimensional Classification Theorem (b)]).

(d) For $m-3\ge 2q\ge 2p$ and $2m\ge 3q+3p+4$ Theorem 1.2 follows by [BG71, Corollary 1.3].

(e) Cf. [LS02] for knotted tori in codimension 1.

Main Theorem 1.3. *(aPL) The group of PL embeddings $S^1\times S^{4k-1}\to\mathbb{R}^{6k+1}$ up to PL isotopy is*

$$KT^{6k+1}_{1,4k-1,PL}\cong\pi^S_{2k-2}\oplus\pi^S_{2k-1}\oplus\mathbb{Z}.$$

(aDIFF) For $k>1$ the group of smooth embeddings $S^1\times S^{4k-1}\to\mathbb{R}^{6k+1}$ up to smooth isotopy is

$$KT^{6k+1}_{1,4k-1,DIFF}\cong\pi^S_{2k-2}\oplus\pi^S_{2k-1}\oplus\mathbb{Z}\oplus G_k,$$

where G_k is an abelian group of order 1, 2 or 4.

(b) The group of PL or smooth embeddings $S^1\times S^{4k+1}\to\mathbb{R}^{6k+4}$ up to PL or smooth isotopy is

$$KT^{6k+4}_{1,4k+1,PL}\cong KT^{6k+4}_{1,4k+1,DIFF}\cong\mathbb{Z}_2^a\oplus\mathbb{Z}_4^b\quad\textit{for some integers } a=a(k),\ b=b(k)\ \textit{such that}$$

$$a+2b-\mathrm{rk}(\pi^S_{2k}\otimes\mathbb{Z}_2)-\mathrm{rk}(\pi^S_{2k-1}\otimes\mathbb{Z}_2)=\begin{cases}0 & k\in\{1,3\}\\ 1 & k+1\ \textit{is not a power of 2}\ .\\ 1\ \textit{or}\ 0 & k+1\ge 8\ \textit{is a power of 2}\end{cases}$$

Remarks. (a) More precisely, for $k>1$ we have $G_k\in\{0,\mathbb{Z}_2,X_k\}$, where $X_k=\mathbb{Z}_2\oplus\mathbb{Z}_2$ for k even and $X_k=\mathbb{Z}_4$ for k odd.

(b) For $l\ge 2$ the forgetful map $KT^{3l+1}_{1,2l-1,AD}\to KT^{3l+1}_{1,2l-1,PL}$ is an isomorphism by the Almost Smoothing Theorem 2.3.

(c) We conjecture that $KT^7_{1,3,DIFF}$ has a geometrically defined group structure and is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_{12}$.

(d) In the Main Theorem 1.3.b $b=b(k)$ is not greater than each of the ranks of $\pi^S_{2k}\otimes\mathbb{Z}_2$ and of $\mathrm{rk}(\pi^S_{2k-1}\otimes\mathbb{Z}_2$.

(e) The Main Theorem 1.3.b for $k+1$ not a power of 2 can be reformulated as follows (corresponding reformulation exists for $k+1$ a power of 2): $KT^{6k+4}_{1,4k+1}\cong\dfrac{\pi_{4k+1}(V_{2k+3,2})}{w_{2k+1,1}}\oplus\mathbb{Z}_2$, *where* $w_{2k+1,1}$ *is defined below.* Hence $KT^{6k+4}_{1,4k+1}\cong\pi_{4k+1}(V_{2k+3,2})$ if $k\not\equiv 3\mod 4$ (by [MS04] for $k>1$ and by §2 for $k=1$). We conjecture that the latter isomorphism holds for each k.

In general there is an exact sequence $0\to\pi^S_{2k}\otimes\mathbb{Z}_2\to\dfrac{\pi_{4k+1}(V_{2k+3,2})}{w_{2k+1,1}}\to\pi^S_{2k-1}\otimes\mathbb{Z}_2\to 0$.

(f) By the Main Theorem 1.3, the above remark and [To62, Pa56, DP97] we have the following table, where u^v means $(\mathbb{Z}_u)^v$ (see the details for $l=5$ in §2).

l	2	3	4	5	6	7	8	9	10
$KT^{3l+1}_{1,2l-1,PL}$	$\mathbb{Z}\oplus\mathbb{Z}\oplus 2$	4	$\mathbb{Z}\oplus 24\oplus 2$	$2\oplus 2$	$\mathbb{Z}$	2	$\mathbb{Z}\oplus 240\oplus 2$	$2\widetilde{\times}2^2$	$\mathbb{Z}\oplus 2^5$

In order to formulate the Main Theorem 1.4 let $S^l\overset{\mu''}{\to}V_{l+p+1,p+1}\overset{\nu''}{\to}V_{l+p+1,p}$ be 'forgetting the last vector' bundle. Denote

$$w_{l,p}=\mu''_*[\iota_l,\iota_l]\in\pi_{2l-1}(V_{l+p+1,p+1}).$$

Denote by $\widehat{\mathbb{Z}_2}$ an unknown group which is either 0 or $\mathbb{Z}_2$, and by $\widehat{\mathbb{Z}}$ an unknown cyclic group. (These groups can vary with the values of l and p.) By $\widehat{2}$ we denote the image of 2 under the quotient map $\mathbb{Z}\to\widehat{\mathbb{Z}}$.

Main Theorem 1.4. *Suppose that* $1\le p\le l-2$.

$$(AD)\qquad KT^{3l+p}_{p,2l-1,AD}\cong\begin{cases}\pi_{2l-1}(V_{l+p+1,p+1}) & l\in\{3,7\}\\ \dfrac{\pi_{2l-1}(V_{l+p+1,p+1})}{w_{l,p}}\oplus\mathbb{Z}_2 & l \textit{ is odd } \neq 2^s-1\\ \dfrac{\pi_{2l-1}(V_{l+p+1,p+1})}{w_{l,p}}\oplus\widehat{\mathbb{Z}_2} & l+1=2^s\ge 16\\ \dfrac{\pi_{2l-1}(V_{l+p+1,p+1})\oplus\mathbb{Z}}{w_{l,p}\oplus 2} & l \textit{ is even}\end{cases}$$

$$(PL)\qquad KT^{3l+p}_{p,2l-1,PL}\cong\begin{cases}\pi_{2l-1}(V_{l+p+1,p+1}) & l\in\{3,7\}\\ \dfrac{\pi_{2l-1}(V_{l+p+1,p+1})}{w_{l,p}}\oplus\widehat{\mathbb{Z}_2} & l \textit{ is odd}\\ \dfrac{\pi_{2l-1}(V_{l+p+1,p+1})\oplus\widehat{\mathbb{Z}}}{w_{l,p}\oplus\widehat{2}} & l \textit{ is even}\end{cases}$$

(DIFF) There is an exact sequence which splits for l odd

$$C^{l+1}_{2l+p-1} \stackrel{\zeta}{\to} KT^{3l+p}_{p,2l-1,DIFF} \stackrel{\text{forg}}{\to} KT^{3l+p}_{p,2l-1,AD} \to 0.$$

Here the map $\zeta : C^{m-p-q}_{p+q} \to KT^m_{p,q,DIFF}$ is defined by setting $\zeta(\varphi)$ to be the embedded connected sum of the standard torus with the embedding $\varphi : S^{p+q} \to \mathbb{R}^m$. The map forg *is the obvious forgetful map.*

Remarks. (a) Since $w_{l,p}$ has infinite order for l even, by the Main Theorem 1.4AD and [DP97] we have

$$\operatorname{rk} KT^{3l+p}_{p,2l-1,AD} = \operatorname{rk} \pi_{2l-1}(V_{l+p+1,p+1}) = \begin{cases} 2 & l = p+1 \text{ is even} \\ 1 & \text{either } l \ge p+2 \text{ is even or } l = p+1 \text{ is odd} \\ 0 & l \text{ is odd and } l \ge p+2 \end{cases}.$$

For l odd the same relation holds also in the PL category; for $2l + p$ not divisible by 4 also in the smooth category. Cf. [CRS].

(b) $KT^{3l+p}_{p,2l-1,PL} \cong KT^{3l+p}_{p,2l-1,AD}/K_{l,p}$, where $K_{l,p}$ is contained in 0, in $0 \oplus \mathbb{Z}_2$, in $0 \oplus \widehat{\mathbb{Z}_2}$ and in the class of $0 \oplus \mathbb{Z}$ for the four cases of the formula for $KT^{3l+p}_{p,2l-1,AD}$, respectively.

(c) In the Main Theorem 1.4 the AD category can be replaced to the category of *smooth* embeddings up to *almost smooth* isotopy (because all representatives of AD embeddings are in fact smooth by the Realization Theorem 2.4 below).

(d) The conditions that $l + 1$ and $k + 1$ are not powers of 2 in the Main Theorems 1.4, 1.3.b can be weakened to $w_{l,p} \neq 0$ and $w_{2k+1,1} \neq 0$, respectively. Note that $w_{l,p} = 0$ for $l \in \{1, 3, 7, 15\}$ (the equality $w_{15,p} = 0$ is due to J. Mukai).

(e) We conjecture that $KT^{3l+p}_{p,2l-1,AD} \cong \pi_{2l-1}(V_{l+p+1,p+1})$ for l distinct from 2, 4 and 8 (with possible exception of those l for which $w_{l,p} = 0$). This is so either if $p = 1$ and $l \equiv 1, 3, 5 \mod 8$ [MS04], or if l even and the projection of $w_{l,p}$ onto the $\mathbb{Z}$-summand of $\pi_{2l-1}(V_{l+p+1,p+1})$ is odd (the latter is so for $p = 1$, l even, $l \ge 6$, $l \neq 8$; note that if the projection of $w_{l,p}$ is even, then $KT^{3l+p}_{p,2l-1,AD}$ is a $\mathbb{Z}_2$-extension of $\pi_{2l-1}(V_{l+p+1,p+1})$).

(f) The Main Theorem 1.4 is a generalization to $p > 0$ of the case $m = 3l$, $q = 2l - 1$ of the Haefliger Theorem 1.1. The formulas of the Main Theorem 1.4 are false for $p = 0$ by the Haefliger Theorem 1.1, and are true but covered by the Main Theorem 1.3 for $p = 1$. Our proof of the Main Theorem 1.4 work with minor modifications to obtain a *new* proof of the Haefliger Theorem 1.1. See details in Remark (13) of §9 and a generalization in [Sk07'].

(g) By [Ha66, 8.15, Mi72, Theorem F and Corollary G] we have $C^{l+1}_{2l+p-1} = 0$ and so $KT^{3l+p}_{p,2l-1,DIFF} \cong KT^{3l+p}_{p,2l-1,AD}$ if

$$\text{either}\quad p = 1,\ l \equiv 1\ mod 2 \quad\text{or}\quad p = 3,\ l \equiv \pm 2\ mod 12 \quad\text{or}\quad p = 5,\ l \equiv 13, 21\ mod 24.$$

(h) For $1 \le p = l - 1$ the Main Theorem 1.4PL is true and the sequence of Main Theorem 1.4.DIFF is an exact sequence of sets with an action ζ, and the map forg has a right inverse.

A counterexample to the Multiple Haefliger-Wu Invariant Conjecture.

For classification of embeddings, as well as in other branches of mathematics, the approach using configuration spaces, or 'complements to diagonals' proved to be very fruitful [Va92]. This approach gives the *Haefliger-Wu invariant* defined as follows. Let

$$\widetilde{N} = \{(x, y) \in N \times N \mid x \neq y\}$$

be the *deleted product* of N, i.e. the configuration space of ordered pairs of distinct points of N. For an embedding $f : N \to \mathbb{R}^m$ one can define a map $\widetilde{f} : \widetilde{N} \to S^{m-1}$ by the Gauss formula

$$\widetilde{f}(x, y) = \frac{fx - fy}{|fx - fy|}.$$

This map is equivariant with respect to the 'exchanging factors' involution $t(x, y) = (y, x)$ on $\widetilde{N}$ and antipodal involution on S^{m-1}. The *Haefliger-Wu invariant* $\alpha(f)$ of the embedding f is the equivariant homotopy class of the map $\widetilde{f}$, cf. [Wu65, Gr86, 2.1.E]. This is clearly an isotopy invariant. Let $\pi^{m-1}_{eq}(\widetilde{N})$ be the set of equivariant maps $\widetilde{N} \to S^{m-1}$ up to equivariant homotopy. Thus the Haefliger-Wu invariant is a map

$$\alpha : \mathrm{Emb}^m(N) \to \pi^{m-1}_{eq}(\widetilde{N}).$$

It is important that using algebraic topology methods the set $\pi^{m-1}_{eq}(\widetilde{N})$ can be explicitly calculated in many cases [Ad93, 7.1, BG71, Ba75, RS99, Sk07]. So it is very interesting to know under which conditions the Haefliger-Wu invariant is bijective, and the bijectivity results we are going to state have many specific corollaries [BG71, Sk07].

Classical Isotopy Theorem 1.5. *(a) [Sk07, the Haefliger-Weber Theorem 5.4] The Haefliger-Wu invariant is bijective for embeddings $N \to \mathbb{R}^m$ of a smooth n-manifold or an n-polyhedron N, if*

$$2m \geq 3n + 4.$$

(b) [Sk07, Theorem 5.5] The Haefliger-Wu invariant is bijective for embeddings $N \to \mathbb{R}^m$ of a d-connected PL n-manifold N, if

$$2m \geq 3n + 3 - d \quad \textit{and} \quad m \geq n + 3.$$

Classical examples of Haefliger and Zeeman [Ha62', §3, RS99, §4, Sk02, Examples 1.2, Sk07, §5] show that the Haefliger-Wu invariant is not bijective without the above metastable dimension assumption $2m \geq 3n + 4$, both in the smooth case and for non-connected N.

(For classical and recent examples of the *non-surjectivity* of the Haefliger-Wu invariant without the restriction $2m \geq 3n + 3$ see [RS99, §4, Sk02, Examples 1.2, MRS03, Theorem 1.3, GS06, Proposition].)

The Whitehead Link Example 1.6. [Sk07, 3.3] *For each l the Haefliger-Wu invariant is not injective for embeddings $S^0 \times S^{2l-1} \to \mathbb{R}^{3l}$, i.e. there exists an embedding $\omega_0 : S^0 \times S^{2l-1} \to \mathbb{R}^{3l}$ which is not isotopic to the standard embedding f_0 but for which $\alpha(\omega_0) = \alpha(f_0)$.*

(The construction of ω_0 is presented at the end of §1.)

On the other hand, the PL Unknotting (and Embedding) Theorems were proved first for $2m \geq 3n + 4$ (or similar restriction) and then for $m \geq n + 3$ [PWZ61, Ze62, Ir65]. Therefore the following conjecture was 'in the air' since 1960's (the author learned it from A. N. Dranishnikov and E. V. Schepin): *the Haefliger-Wu invariant is bijective for PL embeddings of connected PL n-manifolds into $\mathbb{R}^m$ and $m \geq n + 3$.* Even better known is the following refined version of this conjecture (the author learned it from O. Ya. Viro and A. Szücs).

Multiple Haefliger-Wu Invariant Conjecture. *The multiple Haefliger-Wu invariant is bijective for PL embeddings of connected PL n-manifolds into $\mathbb{R}^m$ and $m \geq n + 3$.*

Recall the definition of the multiple Haefliger-Wu invariant [Wu65] (for generalizations see [Wu65, VII, §5, Va92, FM94, RS99, §5, Sk02, §1, Sk07, §5, Me]). Consider the configuration space of i-tuples of pairwise distinct points in N:

$$\widetilde{N}^i = \{(x_1, \dots, x_i) \in N^i \mid x_j \neq x_k \text{ if } \ j \neq k\}.$$

The space $\widetilde{N}^i$ is called the *deleted i-fold product* of N. The group S_i of permutations of i symbols obviously acts on the space $\widetilde{N}^i$. For an embedding $f : N \to \mathbb{R}^m$ define the map

$$\widetilde{f}^i : \widetilde{N}^i \to \widetilde{(\mathbb{R}^m)}^i \quad \text{by} \quad \widetilde{f}^i(x_1, \dots, x_i) = (fx_1, \dots, fx_i).$$

Clearly, the map $\widetilde{f}^i$ is S_i-equivariant. Then we can define the multiple Haefliger-Wu invariant

$$\alpha_\infty = \alpha^m_{\infty,CAT}(N) : \mathrm{Emb}^m_{CAT}(N) \to \oplus_{i=2}^\infty [\widetilde{N}^i, \widetilde{(\mathbb{R}^m)}^i]_{S_i} \quad \text{by} \quad \alpha_\infty(f) = \oplus_{i=2}^\infty [\widetilde{f}^i].$$

Remarks. (a) *The connectedness assumption is essential in the Conjecture* because of the Whitehead Link Example 1.6 and because the example works for multiple Haefliger-Wu invariant (because of the following essentially known results: the Invariance Lemma of §4 and the existence of an almost isotopy Ω_0 from the Whitehead link ω_0 to the standard embedding; the construction of Ω_0 is sketched in the subsection 'almost embeddings and their Haefliger-Wu invariant' of §2 and presented in §4).

(b) *The PL category is also essential in the above conjecture* because in the smooth case the multiple Haefliger-Wu invariant is not injective for $2m < 3n + 4$ and as highly-connected manifold as the sphere S^n [Sk07, §3].

(c) Analogously to the construction of the multiple Haefliger-Wu invariant but using *isovariant* rather than *equivariant* maps one can obtain another invariants. They are possibly stronger but apparently hard to calculate (at least for more complicated manifolds than disjoint unions of spheres; for disjoint union of spheres see Remark (a) to the Haefliger Theorem 1.1). This seems to be one of the reasons why these invariants were not mentioned in [Wu65] (where in the last section even more complicated generalizations were considered).

(d) The Multiple Haefliger-Wu Invariant Conjecture was supported by the fact that analogous invariants were successfully used to classify links or link maps, and to construct new obstructions to embeddability [Ha62', Ma90, Ko91, HK98, Kr00].

(e) Classical Isotopy Theorem 1.5.b shows that the Multiple Haefliger-Wu Invariant conjecture is true under additional $(3n-2m+3)$-connectedness assumption, even for the ordinary (i.e. not multiple) Haefliger-Wu invariant. The inequality $2m \geq 3n + 2 - d$ appeared in the Hudson PL version of the Browder-Haefliger-Casson-Sullivan-Wall Embedding Theorem for closed manifolds (proved by engulfing), but was soon proved to be superfluous (by surgery). So it was natural to expect that the $(3n-2m+3)$-connectivity assumption in Classical Isotopy Theorem 1.5.b is superfluous (Classical Isotopy Theorem 1.5.b was proved using generalization of the engulfing approach). This information again supported the above conjecture.

(f) The restrictions like $2m \geq 3n+3-d$ did appear in results for manifolds with boundary both for PL and smooth cases [Sk02, Theorem 1.1.$\alpha\partial$] (they are sharp by the Filled-Tori Theorem 7.3 below). But such restriction did *not* previously appear for closed manifolds.

(g) By the above remarks it is surprising that the assumption of d-connectedness, not only of connectedness, comes into the dimension range where the Haefliger-Wu invariant is complete: the connectedness assumption is *essential* in Classical Isotopy Theorem 1.5.b [Sk02, Examples 1.4]. This shows that Classical Isotopy Theorem 1.5.b is not quite the result

expected in the 1960's, and that its proof cannot be obtained by direct generalization of the Haefliger-Weber proof without invention of new ideas.

Our next main result disproves the Multiple Haefliger-Wu Invariant Conjecture and shows that in Classical Isotopy Theorem 1.5.b the dimension restriction, or the connectivity assumption, is *sharp* (but not only *essential*).

The Whitehead Torus Example 1.7. *If $l+1$ is not a power of 2, then the (multiple) Haefliger-Wu invariant is not injective for embeddings $S^1 \times S^{2l-1} \to \mathbb{R}^{3l+1}$, i.e. there exists an embedding $\omega_1 : S^1 \times S^{2l-1} \to \mathbb{R}^{3l+1}$ which is not isotopic to the standard embedding f_0 but for which $\alpha(\omega_1) = \alpha(f_0)$ (and even $\alpha_\infty(\omega_1) = \alpha_\infty(f_0)$).*

Remarks. (a) The smooth case of the Whitehead Torus Example 1.7 is not so interesting as the PL case. Indeed, by smoothing theory it is known that $\alpha^{3l+p}_{DIFF}(S^{2l-1+p})$ is not injective for some cases, e.g. $p = 1$ and l even [Ha66', Mi72].

(b) The passage from the Whitehead Link Example 1.6 to the Whitehead Torus Example 1.7 is non-trivial for the following reasons. First, in the Whitehead Link Example 1.6 l is arbitrary, while *the analogue of the Whitehead Torus Example 1.7 is false for $l \in \{3,7\}$* by the Torus Theorem 2.8 and the triviality of the β-invariant Theorem 2.9 below, cf. [Sk05, Isotopy Theorem]. Second, the Whitehead Link Example 1.6 for most l requires only as simple an invariant as the linking coefficient, but a direct analogue of this invariant for tori gives only a weaker example [Sk02, Examples 1.4.i].

The construction (but not the proof!) of the Whitehead Torus Example 1.7 is as simple and explicit as to be given right away. We present here the PL version.

Proof of the Whitehead Torus Example 1.7 is based on this construction and a new embedding invariant (§2,§6). Formally, the Whitehead Torus Example 1.7 follows from the Whitehead Torus Theorem 2.5 and the Almost Smoothing Theorem 2.3 (both below).

Construction of the PL Whitehead link $\omega_{0,l,PL} : S^0 \times S^{2l-1} \to \mathbb{R}^{3l}$. Denote coordinates in $\mathbb{R}^{3l}$ by $(x,y,z) = (x_1,\dots,x_l,y_1,\dots,y_l,z_1,\dots,z_l)$. The Borromean rings is the linking $S^{2l-1}_x \sqcup S^{2l-1}_y \sqcup S^{2l-1}_z \subset \mathbb{R}^{3l}$ of the three spheres given by the equations

$$\begin{cases} x=0 \\ y^2+2z^2=1 \end{cases}, \qquad \begin{cases} y=0 \\ z^2+2x^2=1 \end{cases} \qquad \text{and} \qquad \begin{cases} z=0 \\ x^2+2y^2=1 \end{cases},$$

respectively. Nowthe PL whitehead link is obtained from the Borromean rings by joining the components S^{2l-1}_x and S^{2l-1}_y with a tube.

Construction of the PL Whitehead torus $\omega = \omega_{1,l,PL} : S^1 \times S^{2l-1} \to \mathbb{R}^{3l+1}$. Add a strip to the Whitehead link $\omega_{0,l,PL}$, i.e. extend it to an embedding

$$\omega' : S^0 \times S^{2l-1} \bigcup_{S^0 \times D^{2l-1}_+ = \partial D^1_+ \times D^{2l-1}_+} D^1_+ \times D^{2l-1}_+ \to \mathbb{R}^{3l}.$$

This embedding contains connected sum of the components of the Whitehead link. The union of ω' and the cone over the connected sum forms an embedding $D^1_+ \times S^{2l-1} \to \mathbb{R}^{3l+1}_+$. This latter embedding can clearly be shifted to a proper embedding. The PL Whitehead torus is the union of this proper embedding and its mirror image with respect to $\mathbb{R}^{3l} \subset \mathbb{R}^{3l+1}$.

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2. Main ideas

Notation and conventions.

An embedding $F : N \times I \to \mathbb{R}^m \times I$ is a *concordance* if $N \times 0 = F^{-1}(\mathbb{R}^m \times 0)$ and $N \times 1 = F^{-1}(\mathbb{R}^m \times 1)$. We tacitly use the facts that in codimension at least 3

concordance implies isotopy [Hu70, Li65], and

every concordance or isotopy is ambient [Hu66, Ak69, Ed75].

Clearly, every smooth (i.e. differentiable) map is piecewise differentiable. The forgetful map from the set of piecewise linear embeddings (immersions) up to piecewise linear isotopy (regular homotopy) to the set of piecewise differentiable embeddings (immersions) up to piecewise differentiable isotopy (regular homotopy) is a 1–1 correspondence [Ha67]. Therefore we can consider any smooth map as PL one, although this is incorrect literally.

By $[x]$ we denote the equivalence class of x. However, we often do not distinguish an element and its equivalence class.

Let σ_k^i be the mirror-symmetry of S^k or $\mathbb{R}^k$ with respect to the hyperplane $x_i = 0$. Denote by σ_i the mirror-symmetry with respect to a hyperplane in $\mathbb{R}^i$ or with respect to an equator in S^i, when the concrete choice of the hyperplane or an equator does not play a role. Denote by the same symbol σ_i the map induced by this symmetry on the set of embeddings. An embedding $f : T^{p,q} \to \mathbb{R}^m$ is called *mirror-symmetric* if $\sigma_p f = \sigma_m f$.

Consider the standard decomposition $S^p = D^p_+ \bigcup\limits_{\partial D^p_+ = S^{p-1} = \partial D^p_-} D^p_-$. Analogously define $\mathbb{R}^m_\pm$ and $\mathbb{R}^{m-1}$ by equations $x_1 \ge 0$, $x_1 \le 0$ and $x_1 = 0$, respectively. Denote

$$T^{p,q} := S^p \times S^q \quad \text{and} \quad T^{p,q}_\pm := D^p_\pm \times S^q.$$

Construction of a commutative group structure on $KT^m_{p,q}$.

We denote by 0 the *standard embedding* $T^{p,q} \cong S^q \times S^p \to \mathbb{R}^{q+1} \times \mathbb{R}^{p+1} \subset \mathbb{R}^m$.

A map $f : T^{p,q} \to S^m$ is called *standardized* if

$f(S^p \times D^q_-) \subset D^m_-$ is the restriction of the above standard embedding and

$f(S^p \times \operatorname{Int} D^q_+) \subset \operatorname{Int} D^m_+$.

Roughly speaking, a map $T^{p,q} \to \mathbb{R}^m$ is standardized if its image is put on hyperplane $\mathbb{R}^{m-1}$ so that the image intersects the hyperplane by standardly embedded $S^p \times D^q_-$ (for such a map the image of $S^p \times \operatorname{Int} D^q_-$ can be pulled below the hyperplane to obtain a standardized embedding in the above sense).

A concordance $F : T^{p,q} \times I \to S^m \times I$ between standardized maps is called *standardized* if

$F(S^p \times D^q_- \times I) \subset D^m_- \times I$ is the identical concordance and

$F(S^p \times \operatorname{Int} D^q_+ \times I) \subset \operatorname{Int} D^m_+ \times I$.

Standardization Lemma 2.1. *Assume that*

$m \ge \max\{2p + q + 2, q + 3\}$ in the PL category;

either $m \ge 2p + q + 3$ or $(m, p, q) = (7, 1, 3)$ in the AD category;

$m \ge 2p + q + 3$ in the DIFF category.

Then

any embedding $T^{p,q} \to S^m$ is concordant to a standardized embedding, and

any concordance between standardized embeddings $T^{p,q} \to S^m$ is concordant to a standardized embedding, and any concordance between standardized embeddings $T^{p,q} \to S^m$ is concordant relative to the ends to a standardized concordance.

Let $R_k : \mathbb{R}^k \to \mathbb{R}^k$ be the symmetry with respect to the $(k-2)$-hyperplane given by equations $x_1 = x_2 = 0$.

Group Structure Theorem 2.2. *Under the assumptions of the Standardization Lemma 2.1 for $q \geq 2$ a commutative group structure on $KT^m_{p,q}$ is well-defined by the following construction.*

Let $f_0, f_1 : T^{p,q} \to S^m$ be two embeddings. Take standardized embeddings f_0', f_1' isotopic to them. Let

$$(f_0 + f_1)(x,y) := \begin{cases} f_0'(x,y) & y \in D^q_+ \\ R_m(f_1'(x, R_q y)) & y \in D^q_- \end{cases}.$$

Let $(-f_0)(x,y) := \sigma^2_m f_0'(x, \sigma^2_q y)$. Let 0 be the above standard embedding.

(The image of embedding $f_0 + f_1$ is the union of $f_0'(S^p \times D^q_+)$ and the set obtained from $f_1'(S^p \times D^q_+)$ by axial symmetry with respect to the plane $x_1 = x_2 = 0$.)

Under the assumptions of the Standardization Lemma 2.1 for $q = 1$ we have $\#KT^m_{p,q} = 1$.

The Standardization Lemma 2.1 (except the case $(m,p,q) = (7,1,3)$ in the AD category) and the Group Structure Theorem 2.2 are proved in §3. The proof is a non-trivial generalization of [Ha62, Lemma 1.3 and 1.4].

The following result (proved in §7) reduces the case $(m,p,q) = (7,1,3)$ of the Standardization Lemma 2.1 in the AD category to that in the PL category.

Almost Smoothing Theorem 2.3. *The forgetful map $KT^m_{1,q,AD} \to KT^m_{1,q,PL}$ is a 1–1 correspondence for $m \geq q+4$.*

Explicit construction of generators of $KT^{3l+p}_{p,2l-1}$.

Realization Theorem 2.4.a for $k = 1$. *For smooth embeddings $\tau^1, \tau^2, \omega : T^{1,3} \to \mathbb{R}^7$ constructed below and at the end of §1*

$$KT^7_{1,3,PL} = KT^7_{1,3,AD} = \langle \tau^1, \tau^2, \omega_1 \mid 2\tau^2 = 2\omega_1 \rangle .$$

Construction of τ^1 and τ^2. The maps τ^i are defined as compositions $S^1 \times S^3 \overset{\mathrm{pr}_2 \times t^i}{\to} S^3 \times S^3 \subset \mathbb{R}^7$, where pr_2 is the projection onto the second factor, $\subset$ is the standard inclusion and maps $t^i : S^1 \times S^3 \to S^3$ are defined below. We shall see that $t^i|_{S^1 \times y}$ are embeddings for each $y \in S^3$, hence τ^i are embeddings.

Define $t^1(s,y) := sy$, where S^3 is identified with the set of unit length quaternions and $S^1 \subset S^3$ with the set of unit length complex numbers.

Define $t^2(e^{i\theta}, y) := H(y)\cos\theta + \sin\theta$, where $H : S^3 \to S^2$ is the Hopf map and S^2 is identified with the 2-sphere formed by unit length quaternions of the form $ai + bj + ck$.

The Realization Theorem 2.4.a for $k = 1$ is a particular case of the Realization Theorem 2.3.a below because the above construction of τ^1 and τ^2 is equivalent to the geometric construction below.

Realization Theorem 2.4.a. *Consider the map*

$$\tau^1 \oplus \tau^2 \oplus \omega : \pi^S_{2k-2} \oplus \pi_{4k-1}(S^{2k}) \oplus \mathbb{Z} \to KT^{6k+1}_{1,4k-1,DIFF},$$

where τ^1, τ^2 *and* ω *are defined below and by* $\omega(s) := s\omega_{1,2k,PL}$, *see the end of* §*1. The compositions of this map with the forgetful maps to* $KT^{6k+1}_{1,4k-1,AD}$ *and to* $KT^{6k+1}_{1,4k-1,PL}$ *are epimorphisms with the kernel generated by* $\tau^2[\iota_l,\iota_l]-2\omega(1)$.

Construction of τ^1 *and* τ^2. The maps τ^i are defined as compositions $S^1\times S^{4k-1}\overset{\mathrm{pr}_2\times t^i}{\to} S^{4k-1}\times S^{2k+1}\subset\mathbb{R}^{6k+1}$, where pr_2 is the projection onto the second factor, $\subset$ is the standard inclusion and maps $t^i:S^1\times S^{4k-1}\to S^{4k-1}$ are defined below. We shall see that $t^i|_{S^1\times y}$ are embeddings for each $y\in S^{4k-1}$, hence τ^i are embeddings.

Take any $\varphi\in\pi^S_{2k}\cong\pi_{4k-1}(S^{2k+1})$ realized by a smooth map $\varphi:S^{4k-1}\to S^{2k+1}$. Define t^1 on $S^0\times S^{4k-1}$ by $t^1(\pm1,y)=\pm\varphi(y)$. Take a non-zero vector field on S^{2k+1} whose vectors at each pair of antipodal points are opposite. Using this vector field for each $y\in S^{4k-1}$ we can extend the map $t^1:S^0\times y\to S^{2k+1}$ to a map $t^1:S^1\times y\to S^{2k+1}$.

Take any $\varphi\in\pi_{4k-1}(S^{2k})$ realized by a smooth map $\varphi:S^{4k-1}\to S^{2k}$. Define t^2 on $S^0\times S^{4k-1}$ by $t^2(\pm1,y)=\pm\varphi(y)$. Then for each $y\in S^{4k-1}$ extend the map $t^2:S^0\times y\to S^{2k}$ as a suspension to a map $t^2:S^1\times y\to S^{2k+1}$.

Proof of the Main Theorem 1.3.aPL modulo Realization Theorem 2.4.a. By the Realization Theorem 2.4.a it suffices to prove that $\dfrac{\pi_{2l-1}(V_{l+2,2})\oplus\mathbb{Z}}{w_{l,1}\oplus 2}\cong\pi^S_{l-2}\oplus\pi^S_{l-1}\oplus\mathbb{Z}$ for l even. The isomorphism is probably known, but let us prove it for completeness. For l even the bundle ν'' defined before the Main Theorem 1.4 has a cross-section. Hence $\pi_{2l-1}(V_{l+2,2})\cong\pi^S_{l-2}\oplus\pi_{2l-1}(S^l)$.

Clearly, $w_{l,1}$ goes under this isomorphism to $0\oplus[\iota_l,\iota_l]$. For $l=2,4,8$ by [To62] and the algorithm of finding a quotient of $\mathbb{Z}^t$ over linear relations we have $\dfrac{\pi_{2l-1}(S^l)\oplus\mathbb{Z}}{[\iota_l,\iota_l]\oplus 2}\cong\pi^S_{l-1}\oplus\mathbb{Z}$. For $l\neq 2,4,8$ we have $H[\iota_l,\iota_l]=\pm2$ is the generator of $\operatorname{im}H$. Recall that $\pi_{2l-1}(S^l)\cong\mathbb{Z}\oplus$(a finite group). By the above, the projection of $[\iota_l,\iota_l]$ onto the $\mathbb{Z}$-summand is ±1. Since $\ker(\Sigma:\pi_{2l-1}(S^l)\to\pi^S_{l-1})=\langle[\iota_l,\iota_l]\rangle$, it follows that the above finite group is isomorphic to π^S_{l-1}. Again by the algorithm of finding a quotient of $\mathbb{Z}^t$ over linear relations we obtain that $\dfrac{\pi_{2l-1}(S^l)\oplus\mathbb{Z}}{[\iota_l,\iota_l]\oplus 2}\cong\pi^S_{l-1}\oplus\mathbb{Z}$. $\square$

Realization Theorem 2.4.b. *Let CAT=PL or AD. For* $1\le p\le l-2$ *or* $1=p=l-1$ *there are homomorphisms*

$$\tau^m_{p,q}=\tau_p=\tau:\pi_q(V_{m-q,p+1})\to KT^m_{p,q,DIFF}\quad\text{and}\quad\omega_{p,l}=\omega_p=\omega:\mathbb{Z}_{(l)}\to KT^{3l+p}_{p,2l-1,DIFF}$$

such that for their compositions with the forgetful map $KT^m_{p,q,DIFF}\to KT^m_{p,q,CAT}$, *which compositions are denoted by the same letters* τ *and* ω, *the following holds:*

$\tau^{3l+p}_{p,2l-1}$ *is an isomorphism for* $l\in\{3,7\}$;

$\tau^{3l+p}_{p,2l-1}\oplus\omega_{p,l}$ *is an epimorphism with the kernel generated by* $w_{l,p}\oplus(-2)$ *when* $l+1\neq2^s$ *and CAT=AD;*

$\tau^{3l+p}_{p,2l-1}\oplus\omega_{p,l}$ *is an epimorphism with the kernel generated by* $w_{l,p}\oplus(-2)$ *and* $0\oplus a$ *for some* $a\in\mathbb{Z}_{(l)}$ *when either CAT=PL or* $l+1=2^s\ge16$.

the Main Theorem 1.4 in the AD or PL category follows by the Realization Theorem 2.4.b in the AD or PL category, respectively.

The Realization Theorem 2.4.a in the AD category follows by the case $l+1\neq2^s$ of the Realization Theorem 2.4.b in the AD category because

$$\pi_{4k-1}(V_{2k+2,2})=\pi_{4k-1}(\partial TS^{2k+1})=\pi_{4k-1}(S^{2k})\oplus\pi_{4k-1}(S^{2k+1})\quad\text{and so}\quad\tau^{6k+1}_{1,4k-1}=\tau^1\oplus\tau^2.$$

Here TS^{2k+1} is the space of vectors of length at most 1 tangent to S^{2k+1}.

The Realization Theorem 2.4.a in the PL category follows by the Realization Theorem 2.4.a in the AD category and the Almost Smoothing Theorem 2.3.

Proof of the Main Theorem 1.3.b in the PL category. By the Almost Smoothing Theorem 2.3 it suffices to prove the same result in the AD category. Such a result follows by the Realization Theorem 2.4.b and the following exact sequence, in which $l = 2k + 1$ and $\Sigma\lambda''$ is multiplication by 2 [JW54].

$$\pi_{2l}(S^{l+1}) \underset{\lambda''}{\longrightarrow} \frac{\pi_{2l-1}(S^l)}{[\iota_l, \iota_l]} \underset{\mu''}{\longrightarrow} \frac{\pi_{2l-1}(V_{l+2,2})}{\mu''[\iota_l, \iota_l]} \underset{\nu''}{\longrightarrow} \pi_{2l-1}(S^{l+1}) \underset{\lambda''}{\longrightarrow} \pi_{2l-2}(S^l). \qquad \square$$

Proof of Remark (d) after the Main Theorem 1.3.b for $l = 5$. We need to prove that $\pi_9(V_{7,2})/w_{5,1} \cong \mathbb{Z}_2$. Consider the exact sequence [Pa56, To62]

$$\begin{array}{ccccc} \pi_9(S^5) & \underset{i}{\longrightarrow} & \pi_9(V_{7,2}) & \underset{j}{\longrightarrow} & \pi_9(S^6) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \langle[\iota_5, \iota_5]\rangle = \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_{24} \end{array}$$

Here j could not be a monomorphism, so i is non-zero (which also follows by [Os86]) and we are done. $\square$

The Realization Theorem 2.4.b is proved at the end of §2 using the following construction and the results of the following subsections.

Construction of τ. We follow [Sk02, proof of Torus Lemma 6.1, Sk07, §6]. Recall that $\pi_q(V_{m-q,p+1})$ is isomorphic to the group of smooth maps $S^q \to V_{m-q,p+1}$ up to smooth homotopy. These maps can be considered as smooth maps $\varphi : S^q \times S^p \to \partial D^{m-q}$. Define the smooth embedding $\tau(\varphi)$ as the composition

$$S^p \times S^q \overset{\varphi\times\mathrm{pr}_2}{\to} \partial D^{m-q} \times S^q \subset D^{m-q} \times S^q \subset \mathbb{R}^m.$$

The map $\omega_{p,l}$ is defined by $\omega_{p,l}(s) := s\omega_{p,l,DIFF}$ using Whitehead Torus Theorem 2.5 and the 2-relation of the Relation Theorem 2.7 (both in the next subsection).

The Whitehead torus.

The Whitehead Torus Theorem 2.5. *For each $0 \le p < l$ there is a smooth embedding $\omega_p = \omega_{p,l,DIFF} : T^{p,2l-1} \to \mathbb{R}^{3l+p}$ (the Whitehead torus) such that*

(α-triviality) $\alpha(\omega_p) = \alpha(0)$.

(α_∞-triviality) $\alpha_\infty(\omega_p) = \alpha_\infty(0)$.

(non-triviality) If $l + 1$ is not a power of 2, then ω_p is not almost smoothly isotopic to the standard embedding.

(Note that ω_p is PL isotopic to the standard embedding for $1 \le p < l \in \{3, 7\}$ and is not such for $1 = p < l \ne 2^s - 1$.)

The PL Whitehead link $\omega_{0,l,PL} : T^{0,2l-1} \to \mathbb{R}^{3l}$ is constructed in §1. The almost smooth Whitehead link is the same.

The smooth Whitehead link $\omega_{0,l,DIFF} : T^{0,2l-1} \to \mathbb{R}^{3l}$ is obtained from $\omega_{0,l,PL}$ by connected summation of the second component of $\omega_{0,l,PL}$ with an embedding $\varphi : S^{2l-1} \to S^{3l}$ that represents minus the element of C^{l+1}_{2l-1} obtained by connected summation of the components of $\omega_{0,l,PL}$.

(Note that $\omega_{0,l,PL}$ is smooth but is not smoothly isotopic to $\omega_{0,l,DIFF}$.)

The Whitehead tori $\omega_p = \omega_{p,l,DIFF}$ are constructed inductively from $\omega_0 = \omega_{0,l,DIFF}$ by the Extension Lemma 2.6.a below.

Extension Lemma 2.6. *(a) Let $s \geq q+3$ and $f_0 : T^{0,q} \to \mathbb{R}^s$ be a link. In the smooth case assume that the connected sum of the components of f_0 is a smoothly trivial knot. Then there is a sequence of embeddings $f_p : T^{p,q} \to \mathbb{R}^{s+p}$ such that for $p \geq 1$ embedding f_p is a mirror-symmetric extension of f_{p-1}.*

(b) Suppose that $f_p, g_p : T^{p,q} \to \mathbb{R}^{s+p}$ are two sequences of embeddings such that for $p \geq 1$ embeddings f_p, g_p are mirror-symmetric extensions of f_{p-1}, g_{p-1}, respectively. If f_0 and g_0 are (almost) concordant and $p \leq s-q-2$, then f_p and g_p are (almost) concordant.

Relation Theorem 2.7. *For $1 \leq p \leq l-2$ or $1 = p = l-1$*

(2-relation) $2\omega_p = 0$ in the smooth category for l odd ≥ 3.

(τ-relation) $\tau_p(w_{l,p}) = 2\omega_p$ in the almost smooth category for l even and in the smooth category for l odd.

Note that the 2-relation also holds for $p = 0$.

The properties of the Whitehead torus listed in the Whitehead Torus Theorem 2.5 and the Relation Theorem 2.7 are proved in §4 (using the β-invariant Theorem 2.9 below and explicit constructions). Cf. Symmetry Lemma of §4.

Almost embeddings and their Haefliger-Wu invariant.

The *self-intersection set* of a map $F : N \to \mathbb{R}^m$ is

$$\Sigma(F) = \mathrm{Cl}\{x \in N \mid \#F^{-1}Fx \geq 1\}.$$

A PL map $F : T^{p,q} \to \mathbb{R}^m$ is called a PL *almost embedding*, if $\Sigma(F) \subset T^{p,q}_-$. A map $F : T^{p,q} \to \mathbb{R}^m$ is called a *smooth almost embedding*, if $\Sigma(F) \subset T^{p,q}_-$ and $F|_{T^{p,q}_+}$ is a smooth embedding (i.e. a smooth injective map such that dF_x is non-degenerate at each x). Almost smooth almost embedding is by definition the same as smooth almost embedding.

A proper PL map $F : T^{p,q} \times I \to \mathbb{R}^m \times I$ is called a *PL almost concordance* if $\Sigma(F) \subset T^{p,q}_- \times I$. Analogously a *(almost) smooth almost concordance* is defined. We say that a (almost) concordance F is a (almost) concordance *between* embeddings $F|_{T^{p,q}\times 0}$ and $F|_{T^{p,q}\times 1}$ of $T^{p,q}$ into $\mathbb{R}^m$. The latter two embeddings are called *almost concordant.* (Note that *almost smooth* embedding or concordance is not the same as *smooth almost* embedding or concordance.)

Sketch of a construction of smooth almost isotopy $\Omega_0 : T^{0,1} \times I \to \mathbb{R}^3 \times I$ between the standard link and the Whitehead link. This interesting and well-known example introduces informally one of the main ingredients of our proof (see the details in the non-triviality of the β-invariant Theorem 2.9 below). Recall from §1 that the Whitehead link is obtained from the Borromean rings by joining two of the rings with a tube. It is well-known that after making a transversal self-intersection of the last two of the Borromean rings we can move the first ring away from them. After that, removing the self-intersection by the reverse move we obtain the standard linking. This homotopy is not an almost isotopy. But an almost isotopy is the corresponding homotopy Ω_0 of the Whitehead link, which link is obtained by joining with a tube those two of the Borromean rings that intersect throughout the above homotopy.

Let $\overline{KT}^m_{p,q}$ be the set of almost embeddings $T^{p,q} \to \mathbb{R}^m$ up to almost concordance. For an almost embedding $F : T^{p,q} \to \mathbb{R}^m$ and $m \geq 2p+q+1$ we may assume by general position that $\Sigma(F) \cap T^{p,0} = \emptyset$.

The definition of *standardized almost embedding, standardized almost concordance* are analogous to those at the beginning of §2. In Standardization Lemma 2.1 and Group Structure

Theorem 2.2 (as well as in their proofs) we can replace 'embedding' and 'concordance' by 'almost embedding' and 'almost concordance'.

We have the following diagram:

$$\begin{tikzcd} KT^m_{p,q} \arrow[r, "\mathrm{forg}"'] & \overline{KT}^m_{p,q} \arrow[d, "\bar\alpha"] \\ \pi_q(V_{m-q,p+1}) \arrow[u, "\tau"] \arrow[r, "\alpha\tau=\bar\alpha\,\mathrm{forg}\,\tau"'] & \pi^{m-1}_{eq}(\widetilde{T^{p,q}}) \end{tikzcd}$$

The map forg is the obvious forgetful map. The map α is the Haefliger-Wu invariant defined in §1. The Haefliger-Wu invariant $\bar\alpha$ is well-defined by the Invariance Lemma and proof of α-triviality of §4, cf. [Sk02, Theorem 5.2.α, Sk05, Almost embeddings Theorem]. The map τ is defined above in §2.

Torus Theorem 2.8. *Suppose that*

$$\textit{either}\quad (m,p,q)=(7,1,3)\quad \textit{or}\quad p\le q,\quad m\ge 2p+q+3\quad \textit{and}\quad 2m\ge 3q+p+4.$$

The map $\bar\alpha$ is an isomorphism for

$$2m\ge 3q+2p+2\quad \textit{and CAT=DIFF,}\qquad \textit{or}\qquad 2m\ge 3q+2p+3\quad \textit{and CAT=PL.}$$

The map α is an isomorphism for

$$2m\ge 3q+3p+4\quad \textit{and CAT=DIFF,}\qquad \textit{or}\qquad 2m\ge 3q+2p+4\quad \textit{and CAT=PL.}$$

The map $\alpha\tau$ is an isomorphism for $2m\ge 3q+2p+4$ and is an epimorphism with the kernel $\langle w_{l,p}\rangle$ for $q=2l-1$, $m=3l+p$ and $p\ge 1$.

The assertion on α follows by the Classical Isotopy Theorem 1.5.a,b. The assertion on $\alpha\tau$ follows analogously to [HH62, Sk02, Torus Theorem 6.1], see the details in §5. The assertion on $\bar\alpha$ in the PL category follows by [Sk02, Theorem 2.2.q], cf. [Sk05, Almost Embeddings Theorem (b)]; there is also a direct proof analogous to the proof of the smooth case.

The new part of the Torus Theorem 2.8 is the assertion on $\bar\alpha_{DIFF}$ for $2m\in\{3q+2p+2, 3q+2p+3\}$, which is proved in §5. This assertion implies the bijectivity of the forgetful homomorphism $\overline{KT}^m_{p,q,DIFF}\to\overline{KT}^m_{p,q,PL}$ for $2m=3q+2p+3$. This bijectivity does not follow from the Haefliger smoothing theory [Ha67, Ha] because the smoothing obstructions is in $C^{m-p-q}_q\ne 0$.

(For $2m=3q+2p+2$ the bijectivity of $\overline{\alpha}_{DIFF}$ is not used in the proof of the main results but could be used for an alternative proof of the the Triviality Lemma 6.2, cf. §6.)

(Note that the Almost Smoothing Theorem 2.3 does not follow from the assertion on $\overline{\alpha}$ of the Torus Theorem 2.8: two almost smooth embeddings which are PL isotopic should be smoothly almost isotopic, but not necessarily almost smoothly isotopic.)

(Note that $\alpha^m_{DIFF}(T^{p,q})$ *is surjective for* $2m\ge 3p+3q+3$ [Ha63]. The dimension assumption is sharp here because $\alpha^{3l+p+1}_{DIFF}(T^{p,2l})$ *is not surjective for* $2\le p<l\in\{3,7\}$ (for general manifolds cf. [MRS03, Theorem 1.3]). Indeed, by the Non-triviality Lemma 6.3 below there is $G\in\overline{KT}^{3l+p+1}_{p,2l,DIFF}$ such that $\beta(G)=1$. If $\alpha(F)=\alpha(G)$ for some $F\in KT^{3l+p+1}_{p,2l,DIFF}$, then by the assertion on $\bar\alpha_{DIFF}$ of the Torus Theorem 2.8, F is almost concordant to G. Hence $1=\beta(G)=\beta(F)=0$, which is a contradiction.)

(Note that forg : $KT^m_{p,q,PL} \to \overline{KT}^m_{p,q,PL}$ is an isomorphism for $2m \ge 3q+2p+4$ and is an epimorphism for $2m = 3q+2p+3$.)

A new embedding invariant.

The assertion on $\overline{\alpha}$ of the Torus Theorem 2.8 for $2m = 3q+2p+3$ suggests that in order to classify $KT^{3l+p}_{p,2l-1}$ we need to define for each almost concordance F between embeddings $T^{p,2l-1} \to \mathbb{R}^{3l+p}$ an obstruction $\beta(F)$ to modification of F to a concordance.

Definition of the obstruction $\beta(F)$ for $0 \le p < l$ and an almost embedding $F : T^{p,2l} \to \mathbb{R}^{3l+p+1}$. Take any $x \in D^p_+$. Take a triangulation T of $T^{p,2l}$ such that F is linear on simplices of T and $x \times S^{2l}$ is a subcomplex of T. Then $\Sigma(F)$ is a subcomplex of T. Since F is an almost embedding, we have $\Sigma(F) \subset T^{p,2l}_-$. Denote by

$$[\Sigma(F)] \in C_{l+p-1}(T^{p,2l}_-; \mathbb{Z}_{(l)})$$

the sum of top-dimensional simplices of $\Sigma(F)$.

For l even $[\Sigma(F)]$ is the sum of oriented simplices σ whose $\pm$ coefficients are defined as follows. Fix in advance any orientation of $T^{p,2l}$ and of $\mathbb{R}^{3l+p+1}$. By general position there is a unique simplex σ' of T such that $F(\sigma) = F(\sigma')$. The orientation on σ induces an orientation on $F\sigma$ and then on σ'. The orientations on σ and σ' induce orientations on normal spaces in $T^{p,2l}$ to these simplices. The corresponding orientations in normal spaces in $F(T^{p,2l})$ to $F\sigma$ and to $F\sigma'$ (in this order) together with the orientation on $F\sigma$ induce an orientation on $\mathbb{R}^{3l+p+1}$. If the latter orientation agrees with the fixed orientation of $\mathbb{R}^{3l+p+1}$, then the coefficient of σ is $+1$, otherwise -1. Change of orientation of σ changes all the orientations in the above construction (except the fixed orientations of $T^{p,2l}$ and of $\mathbb{R}^{3l+p+1}$) and so changes the coefficient of σ in $[\Sigma(F)]$. Thus the coefficient is well-defined.

(This definition of signs is equivalent to that of [Hu69, §11, Hu70', cf. Sk07, §2] given as follows. The orientation on σ induces an orientation on $F\sigma$ and on σ', hence it induces an orientation on their links. Consider the oriented $(2l+1)$-sphere $\operatorname{lk} F\sigma$, that is the link of $F\sigma$ in certain triangulation of $\mathbb{R}^{3l+p+1}$ 'compatible' with T. This sphere contains disjoint oriented l-spheres $F(\operatorname{lk}_T \sigma)$ and $F(\operatorname{lk}_T \sigma')$. Their linking coefficient $\operatorname{link}_{\operatorname{lk} F\sigma}(F(\operatorname{lk}_T \sigma), F(\operatorname{lk}_T \sigma')) \in \mathbb{Z}_{(l)}$ is the coefficient of σ in $[\Sigma(F)]$, which equals ± 1. For l odd these signs can also be defined but are not used because $\partial[\Sigma(F)] = 0$ only mod2.)

We have $\partial[\Sigma(F)] = 0$ (only mod2 if l is odd) [Hu69, Lemma 11.4, Hu70', Lemma 1, cf. Sk07, §2, the Whitney obstruction]. Since

$$\Sigma(F) \subset T^{p,2l}_- \quad \text{and} \quad l+p-1 < 2l, \quad \text{we have} \quad [\Sigma(F)] = \partial C \quad \text{for some} \quad C \in C_{l+p}(T^{p,2l}_-; \mathbb{Z}_{(l)}).$$

We have $\partial FC = 0$ [Hu70', Corollary 1.1]. Hence

$$FC = \partial D \quad \text{for some} \quad D \in C_{l+p+1}(\mathbb{R}^{3l+p+1}; \mathbb{Z}_{(l)}).$$

Since the support of C is in $T^{p,2l}_-$, it follows that $FC \cap F(x \times S^{2l}) = \emptyset$. So define

$$\beta(F) := D \cap [F(x \times S^{2l})] \in \mathbb{Z}_{(l)},$$

where $\cap$ denotes algebraic intersection.

Analogously one constructs *the obstruction $\beta(F) \in \mathbb{Z}_{(l)}$ for an almost concordance F between embeddings $T^{p,2l-1} \to \mathbb{R}^{3l+p}$ almost concordant to the standard embedding.* Only replace

$$T^{p,2l} \quad \text{by} \quad T^{p,2l-1} \times I, \quad T^{p,2l}_\pm \quad \text{by} \quad T^{p,2l-1}_\pm \times I \quad \text{and} \quad \mathbb{R}^{3l+p+1} \quad \text{by} \quad \mathbb{R}^{3l+p} \times I.$$

(The condition that the 'boundary embeddings' of F are almost concordant to the standard embedding is used not for the definition, but for the proof of the independence on C in the case $p = l - 1$, which case is only required for the Main Theorem 1.3.aPL for $k = 1$ and Remark (h) after the Main Theorem 1.4.)

Proof that β-obstruction is well-defined, i.e. is independent on the choices C and D. The independence on the choice of D is standard.

In order to prove the independence on the choice of C assume that F is an almost embedding and $\partial C_1 = \partial C_2 = [\Sigma(F)]$. Then

$$\partial(C_1 - C_2) = 0 \quad \text{and} \quad C_1 - C_2 \in C_{l+p}(T^{p,2l}_-; \mathbb{Z}_{(l)}).$$

Since

$$l + p < 2l, \quad \text{we have} \quad C_1 - C_2 = \partial X \quad \text{for some} \quad X \in C_{l+p+1}(T^{p,2l}_-; \mathbb{Z}_{(l)}).$$

Thus $FC_1 - FC_2 = \partial FX$. Hence we can take D_1 and D_2 so that $D_1 - D_2 = FX$ is disjoint with $[F(x \times S^{2l})]$. Therefore $D_1 \cap [F(x \times S^{2l})] = D_2 \cap [F(x \times S^{2l})]$.

For an almost concordance F the proof is analogous when $p \le l - 2$. When $p = l - 1$ we have $\dim C = 2l - 1$. So $C_1 - C_2$ is homologous to $s[\{-x\} \times S^{2l-1} \times 0]$ in $T^{p,2l-1}_-$ for some integer s. Hence $F(C_1 - C_2)$ is homologous to $sF[\{-x\} \times S^{2l-1} \times 0]$. Since F is an almost concordance between embeddings almost concordant to the standard embedding, it follows that F extends to an almost embedding $\bar F : S^p \times D^{2l}_+ \to D^{3l+p+1}_+$. Since $C_1 - C_2$ is null-homologous in $D^p_- \times D^{2l}_+$, it follows that $F(C_1 - C_2)$ is null-homologous in $D^{3l+p+1}_+ - \bar F(x \times D^{2l}_+)$ and hence in $S^{3l+p} \times I - F(x \times S^{2l-1} \times I)$. $\square$

The constructed β-obstruction is similar not only to [Hu70', Ha84] but also to the Fenn-Rolfsen-Koschorke-Kirk β-invariant of link maps [Ko88, §4] and to the Sato-Levine invariant of semi-boundary links (for the Kirk-Livingston-Polyak-Viro-Akhmetiev generalized Sato-Levine invariant see [MR05]).

A map $f : T^{0,q} \to \mathbb{R}^m$ is an almost embedding if the images of the components are disjoint and the restriction to the second component is an embedding. Thus for $p = 0$ the above $\beta(F : T^{p,2l} \to \mathbb{R}^{3l+p+1})$ is a PL version of the Koschorke β-invariant [Ko88, §4, Ko90, §2]. For $p = 0$ and $l = 1$ the above $\beta(F : T^{p,2l-1} \times I \to \mathbb{R}^{3l+p} \times I)$ is the Sato-Levine invariant. (It is independent on the choice of C for links almost concordant to the standard link, but not for arbitrary links.)

Properties of the new embedding invariant.

β-invariant Theorem 2.9. *Suppose that $0 \le p < l$ and F is an almost concordance between embeddings $T^{p,2l-1} \to \mathbb{R}^{3l+p}$ almost concordant to the standard embedding. We assume PL, AD or DIFF category, except PL in the independence and DIFF in the completeness.*

(obstruction) If F is an embedding, then $\beta(F) = 0$.

(invariance) $\beta(F)$ is invariant under almost concordance of F relative to the boundary.

(completeness) If $\beta(F) = 0$ and $l \ge 2$, then F is almost concordant relative to $T^{p,2l-1}_+ \times I \cup T^{p,2l-1} \times \{0,1\}$ to a concordance.

(independence) If $l+1$ is not a power of 2, then for smooth almost concordances F between embeddings $f_0, f_1 : T^{p,2l-1} \to \mathbb{R}^{3l+p}$, the class $\beta(F)$ depends only on f_0 and f_1.

(triviality) If $1 \le p < l \in \{3, 7\}$, then there is a concordance F_1 such that $F_1 = F$ on $T^{p,2l-1}_+ \times I$ and F_1 is a concordance between the same embeddings as the almost concordance F.

(non-triviality) There exists a smooth almost concordance Ω_p between the Whitehead torus $\omega_p : T^{p,2l-1} \to B^{3l+p}$ and the standard embedding such that $\beta(\Omega_p) = 1$.

The obstruction follows obviously by the definition of $\beta(F)$. The invariance is simple and analogous to [Hu70', Lemma 2, cf. Hu69, Lemma 11.6], see the details in [Sk05]. The completeness is a non-trivial property, but it is easily implied by known results [Hu70', Theorem 2, Ha84, Theorem 4], see the details in §6.

The remaining properties are new and non-trivial.

The non-triviality is proved in §4 using a higher-dimensional analogue of the above example of an almost concordance Ω_0 and the Extension Lemma 2.6.b.

The independence and the triviality are proved in §6. In their proof we use an analogue of the Koschorke formula stating that for link maps the β-invariant is the composition of the α-invariant and the Hopf homomorphism. The triviality implies that *the independence is false for $l \in \{3,7\}$*, thus the proof of the independence should not work for $l \in \{3,7\}$ and so could not be easy. Note that the idea of the proof of a property analogous to the independence [HM87] does not work in our situation.

Note that the independence holds in the PL category for $p = 1$ by the Almost Smoothing Theorem 2.3. The condition that $l+1$ is not a power of 2 in the independence (also in its proof and applications) can be relaxed by $w_{l,p} \neq 0$.

Let us state another simple properties of β-invariant. Denote by $\overline{F}$ the reversed F, i.e. $\overline{F}(x,t) = F(x, 1-t)$. For almost concordances F, F' between f_0 and f_1, f_1 and f_2, respectively, define an almost concordance $F \cup F'$ between f_0 and f_2 as 'the union' of

$$F : X \times [0,1] \to \mathbb{R}^m \times [0,1] \quad \text{and} \quad F' : X \times [1,2] \to \mathbb{R}^m \times [1,2].$$

(Observe that the union of almost concordances is associative up to ambient isotopy.)

Analogously to the Group Structure Theorem 2.2 for $m \geq 2p+q+3$ in the DIFF or AD category and for $m \geq \max\{2p+q+2, q+3\}$ in the PL category, we can construct a sum operation (not necessarily well-defined) on the set of almost concordances $T^{p,q} \times I \to \mathbb{R}^m \times I$, extending the sum operation on the boundary.

(We can prove that the sum operation is well-defined only for $m \geq 2p+q+4$ in the DIFF or AD category and for $m \geq 2p+q+3$ in the PL category, because the analogue of the Standardization Lemma 2.1 for almost concordances between almost concordances would require such assumptions).

Additivity Theorem 2.10. *Suppose that $0 \leq p < l$ and F, F' are almost concordances between embeddings $T^{p,2l-1} \to \mathbb{R}^{3l+p}$ almost concordant to the standard embedding.*

(a) $\beta(F \cup F') = \beta(F) + \beta(F')$ and $\beta(\overline{F}) = -\beta(F)$.

(b) $\beta(F + F') = \beta(F) + \beta(F')$ (for $p = l-1$ only in the PL category).

Part (a) follows obviously by the definition of $\beta(F)$. Part (b) is simple, see the details in §6.

Proof of the Realization Theorem 2.4.b.

In this subsection we omit AD category from the notation (but in this subsection this does not mean that the same holds for PL or DIFF category). Denote $\Pi_{l,p} = \pi_{2l-1}(V_{l+p+1,p+1})$.

Proof of the Realization Theorem 2.4.b in the AD category. By the assertion on $\alpha\tau$ of the Torus Theorem 2.8 we can identify the groups $\pi^{3l+p-1}_{eq}(\widetilde{T^{p,2l-1}}) \cong \Pi_{l,p}/w_{l,p}$ so that $\alpha\tau$ is identified with $\mathrm{pr} : \Pi_{l,p} \to \Pi_{l,p}/w_{l,p}$.

First assume that $l \in \{3,7\}$. Then $\alpha\tau = \pm\,\mathrm{id}\,\Pi_{l,p}$, so it remains to prove that $\pm\tau\alpha(f) = f$ for each $f \in KT^{3l+p}_{p,2l-1}$. Since $\alpha\tau\alpha(f) = \pm\alpha(f)$, by the assertion on $\bar\alpha$ of the Torus Theorem 2.8

there is an almost concordance F between $\pm\tau\alpha(f)$ and f. By the triviality of the β-invariant Theorem 2.9 we have $\pm\tau\alpha(f) = f$.

Now assume that $l \notin \{3, 7\}$. Consider the following diagram. We omit indices p and l from the notation of $\omega_{l,p}$ and $\tau_{l,p}$ in this proof.

$$\begin{tikzcd}
\ker\omega \arrow[r] & \mathbb{Z}_{(l)} \arrow[r, "\omega"'] \arrow[dr, "i"'] & KT^{3l+p}_{p,2l-1} \arrow[r, "\alpha"'] & \Pi_{l,p}/w_{l,p} \arrow[r] & 0 \\
& & \dfrac{\Pi_{l,p} \oplus \mathbb{Z}_{(l)}}{w_{l,p} \oplus (2\mathbb{Z}_{(l)} + \ker\omega)} \arrow[u, "\widehat{\tau\oplus\omega}"] \arrow[ur, "j"'] & &
\end{tikzcd}$$

Here $i(x) := [(0, x)]$ and $j[(a, b)] := [a]$ (clearly, j is well-defined by this formula). Recall the construction of the maps τ and ω after the Realization Theorem 2.4.b. By the Relation Theorem 2.7 we can define the map $\widehat{\tau \oplus \omega}$ by $[a \oplus b] \mapsto \tau(a) + b\omega$. Clearly, the left triangle is commutative. By the α-triviality of the Whitehead Torus Theorem 2.5, $\alpha\omega = 0$. This and $\alpha\tau = \mathrm{pr}$ imply that the right triangle is commutative. Since $\alpha\tau = \mathrm{pr}$, it follows that α is epimorphic, i.e. the first line is exact at $\Pi_{l,p}/w_{l,p}$. The sequence is also exact at $\mathbb{Z}_{(l)}$. So if we prove that $\ker\alpha \subset \operatorname{im}\omega$, then the 5-lemma would imply that $\widehat{\tau \oplus \omega}$ is an isomorphism. Then the Theorem would follow because by the non-triviality of the Whitehead Torus Theorem 2.5 $\ker\omega = 0$ for $l + 1 \neq 2^s$. $\square$

Proof that $\ker\alpha \subset \operatorname{im}\omega$. Suppose that $\alpha(f) = 0$ for some $f \in KT^{3l+p}_{p,2l-1}$. Then by the assertion on $\bar\alpha$ of the Torus Theorem 2.8 there is an almost concordance F between f and the standard embedding.

Take an almost concordance Ω given by the non-triviality of the β-invariant Theorem 2.9. Let $\beta(F)\Omega$ be the sum of $\beta(F)$ copies of Ω (recall that for $p = l - 1$ the sum is not necessarily well-defined, but we just take *any* of the possible sums). Then $\beta(F)\Omega \cup \overline{F}$ is an almost concordance between $\beta(F)\omega$ and f. By the Additivity Theorem 2.10 and the non-triviality of the β-invariant Theorem 2.9, $\beta(\beta(F)\Omega \cup \overline{F}) = \beta(F)\beta(\Omega) - \beta(F) = 0$. Hence by the completeness of the β-invariant Theorem 2.9, $\beta(F)\Omega \cup \overline{F}$ is almost concordant relative to the boundary to a concordance. So $f = \beta(F)\omega \in \operatorname{im}\omega$. $\square$

Proof of the Realization Theorem 2.4.b in the PL category. For $l \in \{3, 7\}$ the proof is the same as in the AD category. For $l \notin \{3, 7\}$ analogously to the AD category we obtain an isomorphism

$$\tau_{PL}\widehat{\oplus\,\omega}_{PL} : \frac{\Pi_{l,p} \oplus \mathbb{Z}_{(l)}}{w_{l,p} \oplus (2\mathbb{Z}_{(l)} + \ker\omega_{PL})} \to KT^{3l+p}_{p,2l-1,PL}. \qquad \square$$

(For $p \geq 2$ we do not know that $\ker\omega_{PL} = 0$ for $l + 1 \neq 2^s$. Note that the forgetful map $KT^{3l+p}_{p,2l-1,AD} \to KT^{3l+p}_{p,2l-1,PL}$ is surjective and its kernel is $\ker\omega_{PL}/\ker\omega_{AD}$.)

3. Proof of the Group Structure Theorem 2.2

Fix a point $y \in D^q_- \subset S^q$.

Proof of the Standardization Lemma 2.1 for embeddings in the PL category. Since $m \geq 2p + q + 2 \geq 2p + 2$, it follows that $g|_{S^p \times y}$ is unknotted in S^m. So there is an embedding $\hat g : D^{p+1} \subset S^m$ such that

$$(1) \qquad \hat g|_{\partial D^{p+1}} = g|_{S^p \times y}.$$

Since $m \geq 2p+q+2$, by general position we may assume that

$$(2) \qquad \hat g \operatorname{Int} D^{p+1} \cap gT^{p,q} = \emptyset.$$

The regular neighborhood in S^m of $\hat g D^{p+1}$ is homeomorphic to the m-ball. Take an isotopy moving this ball to D^m_- and let $f' : T^{p,q} \to S^m$ be the PL embedding obtained from g.

Now we are done since the embedding $f'(S^p \times D^q_-) \subset D^m_-$ is PL isotopic to the standard embedding by the following result (because $m \geq p+q+3$, the pair $(S^p \times D^q, S^p \times S^{q-1})$ is $(q-1)$-connected and $q-1 \geq 2(p+q)-m+1$). $\square$

Unknotting Theorem Moving the Boundary. *Let N be a compact n-dimensional PL manifold and $f, g : N \to D^m$ proper PL embeddings. If $m \geq n+3$ and $(N, \partial N)$ is $(2n-m+1)$-connected, then f and g are properly isotopic* [Hu69, Theorem 10.2].

Proof of the Standardization Lemma 2.1 for embeddings in the smooth category. The bundle $\nu_{S^m}(g|_{S^p \times y})$ is stably trivial and $m-p-q \geq p$, hence this bundle is trivial. Take a $(m-p-q)$-framing ξ of this bundle.

Take the section formed by the first vectors of ξ. As for the PL category we can 'extend' this section to a smooth embedding $\widehat g : D^{p+1} \to \mathbb{R}^m$ satisfying to (1) and (2). Take an isotopy moving the regular neighborhood in S^m of $\widehat g D^{p+1}$ to D^m_- and let $f' : T^{p,q} \to S^m$ be the embedding obtained from g by this isotopy.

By deleting the first vector from ξ we obtain a $(m-p-q-1)$-framing ξ_1 on $\widehat g(\partial D^{p+1})$ normal to $\widehat g(D^{p+1})$. Denote by η the standard normal q-framing of $g(S^p \times y)$ in $gT^{p,q}$. Then (ξ_1, η) is a normal $(m-p-1)$-framing on $\widehat g(\partial D^{p+1})$ normal to $\widehat g(D^{p+1})$. Since $p < m-p-q-1$, the map $\pi_p(SO_{m-p-q-1}) \to \pi_p(SO_{m-p-1})$ is epimorphic. Hence we can change ξ_1 (and thus ξ) so that (ξ_1, η) extends to a normal framing on $\widehat g(D^{p+1})$. Hence $f'(S^p \times D^q_-) \subset D^m_-$ is isotopic to the standard embedding. $\square$

Proof of the Standardization Lemma 2.1 for concordances in the PL category. This is a relative version of the proof for embeddings with some additional efforts required for $m = 2p+q+2$. Take a concordance G between standardized embeddings $f_0, f_1 : T^{p,q} \to S^m$. There is a level-preserving embedding $\hat G : D^{p+1} \times \{0,1\} \to S^m \times \{0,1\}$ whose components satisfy to (1) and (2). Since $m \geq 2p+q+2$ and $m \geq q+3$, it follows that $m+1 \geq p+1+3$. Hence $\hat G$ can be extended to an embedding $\hat G : D^{p+1} \times I \subset S^m \times I$ such that

$$(1') \qquad \hat G|_{\partial D^{p+1} \times I} = G|_{S^p \times y \times I}.$$

Indeed, it suffices to see that any concordance $S^p \times I \to S^m \times I$ standard on the boundary is isotopic to the standard concordance, which follows from the Haefliger-Zeeman Unknotting Theorem.

If $m \geq 2p+q+3$, then by general position

$$(2') \qquad \hat G(\operatorname{Int} D^{p+1} \times I) \cap G(T^{p,q} \times I) = \emptyset.$$

If $m = 2p+q+2$ (in particular, $(m,p,q) = (7,1,3)$), then $p \geq 1$. So either $q = 0$ and the Lemma is obvious or the property (2') can be achieved analogously to the Proposition below because $m+1 = 2(p+1)+q+1$, $p+1 \geq 2$ and $q \geq 1$.

Take a regular neighborhood

$$B^m \times I \quad \text{in} \quad S^m \times I \quad \text{of} \quad \hat G D^{p+1} \quad \text{such that} \quad (B^m \times I) \cap (S^m \times \{0,1\}) = D^m_- \times \{0,1\}.$$

Take an isotopy of $S^m \times I$ rel $S^m \times \{0,1\}$ moving $B^m \times I$ to $D^m_- \times I$. Let F' be the concordance obtained from G by this isotopy.

The embedding $F'(S^p \times D^q_- \times I) \subset D^m_- \times I$ is isotopic $\operatorname{rel} D^m_- \times \{0,1\})$ to the identical concordance by the following Unknotting Theorem Moving Part of the Boundary (which is proved analogously to [Hu69, Theorem 10.2 on p. 199]).

Let N be a compact n-dimensional PL manifold, A a codimension zero submanifold of ∂N and $f,g : N \to D^m$ proper PL embeddings. If $m \ge n+3$ and (N,A) is $(2n-m+1)$-connected, then f and g are properly isotopic $\operatorname{rel} \partial N - A$. $\square$

Proposition. *If $m = 2p+q+1$, $p \ge 2$, $q \ge 1$ and $g : T^{p,q} \to S^m$ is a PL embedding, then there is a PL embedding $\hat g : D^{p+1} \to S^m$ such that (1) and (2) hold.*

Proof. Since $m = 2p+q+1 \ge p+q+3$, it follows that $2m \ge 3p+4$. Hence $g(S^p \times y)$ is unknotted in S^m. So there is an embedding $g' : D^{p+1} \subset S^m$ such that $g'|_{\partial D^{p+1}} = g|_{S^p \times y}$. We shall achieve (2) by modification of D^{p+1} modulo the boundary analogously to [Hu63, proof of Lemma 1 for $M =$ a point, cf. Sk98].

By general position $g' \operatorname{Int} D^{p+1} \cap gT^{p,q}$ is a finite number of points. These points can be joined by an arc in $g'D^{p+1}$ meeting ∂D^{p+1} only in endpoint x. These points can also be joined by an arc in $gT^{p,q}$ meeting $g(S^p \times y)$ in the endpoint x only and passing the double ponts in the same order as the previous arc. The union of these two arcs is contained in a union $C^2 \subset S^m$ of 2-disks. Since $m \ge p+4$ and $m \ge p+q+3$, by general position it follows that $C^2 \cap D^{p+1}$ and $C^2 \cap gT^{p,q}$ are exactly the first and the second of the above arcs, respectively. A small regular neighborhood of C^2 in $\mathbb{R}^m$ is a ball B^m. Therefore

$B^m \cap gT^{p,q}$ is a proper $(p+q)$-ball in B^m;

$B^m \cap \partial D^{p+1}$ is a proper p-ball in the ball $B^m \cap gT^{p,q}$;

$B^m \cap D^{p+1}$ is a $(p+1)$-ball in B^m whose boundary is contained in $\partial D^{p+1} \cup \partial B^m$; and

$\partial B^m \cap D^{p+1} \cap gT^{p,q} = \partial B^m \cap \partial D^{p+1}$.

Since $m \ge p+q+3$, by the Unknotting Balls Theorem there is a PL homeomorphism $h : \operatorname{con} \partial B^m \to B^m$ identical on the boundary and such that $h \operatorname{con}(\partial B^m \cap gT^{p,q}) = B^m \cap gT^{p,q}$. Then replacing $B^m \cap g'D^{p+1}$ with $h \operatorname{con}(\partial B^m \cap g'D^{p+1})$ we obtain the required embedding $\hat g$. $\square$

The Standardization Lemma 2.1 for concordances in the smooth category is proved using the ideas of proof of the Standardization Lemma 2.1 for concordances in the PL category and for embeddings in the smooth category.

The Standardization Lemma 2.1 in the AD category is proved analogously to the one in the smooth category (all the modifications are performed outside a ball on which the embedding or the concordance is not smooth).

Proof of the Group Structure Theorem 2.2. By the Standardization Lemma 2.1 for concordances, the concordance (= the isotopy) class of $f_0 + f_1$ does not depend on the choice of f_0' and f_1'.

Clearly, $f + 0 = f$.

Let $R_{k,t}$ be the rotation of $\mathbb{R}^k$ whose restriction to the plane $\mathbb{R}^2$ generated by e_1 and e_2 is a rotation through the angle πt and which leaves the orthogonal complement $\mathbb{R}^{k-2}$ fixed. For each embedding $f : T^{p,q} \to \mathbb{R}^m$ the embeddings f and $R_{m,-t} \circ f \circ (\mathrm{id}_{S^p} \times R_{q,t})$ are isotopic. The sum operation is *commutative* because

$$f_0' + f_1' \quad \text{is isotopic to} \quad R_{m,1} \circ (f_0' + f_1') \circ (\mathrm{id}_{S^p} \times R_{q,1}) = f_1' + f_0'.$$

In order to prove *the associativity*, denote $D^q_{++} = \{x \in D^q \mid x_1 \ge 0,\ x_2 \ge 0\}$ and define $\mathbb{R}^m_{++}$, D^q_{+-}, D^q_{-+} and D^q_{--} analogously. By the Standardization Lemma 2.1 *any embedding $T^{p,q} \to \mathbb{R}^m$ is isotopic to an embedding $f : T^{p,q} \to \mathbb{R}^m$ such that*

$f : S^p \times (S^q - \operatorname{Int} D^q_{++}) \to \mathbb{R}^m - \operatorname{Int} \mathbb{R}^m_{++}$ is the standard embedding and

$f(S^p \times \operatorname{Int} D^q_{++}) \subset \operatorname{Int} \mathbb{R}^m_{++}$.

If f, g and h are such embeddings, then both $f + (g + h)$ and $(f + g) + h$ are isotopic to the embedding which is standard on $S^p \times D^q_{--}$ and whose restrictions to $S^p \times \operatorname{Int} D^q_{+-}$, to $S^p \times \operatorname{Int} D^q_{++}$ and to $S^p \times \operatorname{Int} D^q_{-+}$ are obtained by rotation from the restrictions of f, g and h onto $S^p \times \mathring{D}^q_{++}$.

Clearly, the embedding $-f_0'$ is standardized. The embedding $f_0' + (-f_0')$ can be extended to an embedding $S^p \times D^{q+1}_+ \to \mathbb{R}^{m+1}_+$ by mapping linearly the segment $x \times [y, \sigma^1_q y]$ to the segment $fx \times [fy, \sigma^1_m fy]$. Hence the embedding $f_0' + (-f_0')$ is isotopic to the standard one by the Triviality Criterion below. $\square$

Triviality Criterion. *For $m \ge \max\{2p+q+2, q+3\}$ an embedding $f : T^{p,q} \to S^m$ is isotopic to the standard embedding if and only if it extends to an embedding $\overline{f} : S^p \times D^{q+1}_+ \to D^{m+1}_+$.*

Proof. The 'only if' part is easy and does not require the dimension assumption. In order to prove the 'if' part observe that analogously to the Standardization Lemma 2.1

for $m \ge \max\{2p+q+2, q+3\}$ any proper embedding $S^p \times D^{q+1}_+ \to D^{m+1}_+$ is concordant relative to the boundary to an embedding $f : S^p \times D^{q+1}_+ \to S^{m+1}$ such that

$f(S^p \times \frac12 D^{q+1}_+) \subset \frac12 D^{m+1}_+$ is the standard embedding and

$f(S^p \times (D^{q+1}_+ - \frac12 D^{q+1}_+)) \subset D^{m+1}_+ - \frac12 D^{m+1}_+$.

Apply this assertion to $\overline{f}$. Then the restriction of the obtained embedding to $S^p \times (D^{q+1}_+ - \frac12 \operatorname{Int} D^{q+1}_+) \to D^{m+1}_+ - \frac12 \operatorname{Int} D^{m+1}_+$ is a concordance from f to the standard embedding. $\square$

Remarks. (a) *For $p+1 = q = 2k$ and $m = 2p+q+2 = 6k$ there are no group structures on $KT^m_{p,q,DIFF}$ such that the Whitney invariant $W : KT^m_{p,q,DIFF} \to \mathbb{Z}$ [Sk07, Sk06] is a homomorphism.* Indeed, W-preimages of distinct elements consist of different number of elements [Sk06, Classification Theorem and Higher-dimensional Classification Theorem]. (Note that $W = \alpha$.)

For $p = q = 2k$ and $m = 2p+q+1 = 6k+1$ there are no group structures on $KT^m_{p,q,DIFF}$ such that the Whitney invariant $W : KT^m_{p,q,DIFF} \to \mathbb{Z}\oplus\mathbb{Z}$ [BH70, KS05] is a homomorphism (even a sum operation which is not well-defined does not exist). Indeed, $W(KT^{6k+1}_{2k,2k,DIFF}) = 2\mathbb{Z}\oplus 0 \cup 0 \oplus 2\mathbb{Z}$ [Bo71] which is not a subgroup of $\mathbb{Z}\oplus\mathbb{Z}$. (Note that $BH = 2W = 2\alpha$.)

(b) In the Standardization Lemma 2.1 for embeddings in the DIFF and AD categories we only need $m \ge \max\{2p+q+2, q+3\}$. Hence there is a sum operation '+' on $KT^m_{p,q,DIFF}$ which is not necessarily well-defined. It is indeed not well-defined on $KT^{6k}_{2k-1,2k,DIFF}$ because for the Whitney invariant W [Sk07, Sk06] we have $W(f+g) = W(f) + W(g)$ and W-preimages of distinct elements consist of different number of elements

(c) *The analogues for $m = 2p+q+1$ of the Standardization Lemma 2.1 for embeddings and of the Triviality Criterion are false in the PL category*, in spite of the Proposition. Indeed, a proper embedding $S^p \times D^q \subset D^{2p+q+1}$ is not necessarily properly isotopic to the standard embedding; counterexamples to the analogue for $m = 2p+q+1 \ge 2q+p+2$ of the Triviality Criterion are obtained by taking a non-trivial embedding $T^{p,q} \to \partial B^{2p+q+1}$ and then extending them to an embedding $S^p \times D^{q+1} \to B^{2p+q+2}$ analogously to §4 or to [Hu63].

(d) The Standardization Lemma 2.1 in the smooth category can also be proved analogously to the PL category using the smooth analogue of Unknotting Theorem Moving the Boundary (for which no additonal metastable dimension restriction is required). The PL case can in turn be proved analogously to the smooth case, we only replace SO by PL, use the Fibering Lemma 7.1 below and $\pi_p(V^{PL}_{m-q,p+1}) = 0$ for $m \ge 2p+q+2$ [HW66]; instead of k-frames over X we consider PL embeddings of $X \times D^k$.

(e) *An alternative proof of the if part of the Triviality Criterion in the PL category without use of the analogue of the Standardization Lemma 2.1.* Take an extension $\overline{f}_0 : S^p \times D^{q+1}_+ \to B^{m+1}_+$ of the standard embedding $f_0 : T^{p,q} \to S^m$. Since $m \geq 2p + q + 2$, by the Unknotting Theorem Moving the Boundary any two proper embeddings $S^p \times D^q_- \to B^m_-$ are properly isotopic. Hence $\overline{f}$ is properly isotopic to $\overline{f}_0$, so f is isotopic to f_0.

(f) Under conditions of the Standardization Lemma 2.1 for embeddings the following interesting situation occur: any proper embedding $S^p \times D^q_\pm \to D^m_\pm$ is isotopic to the standard embedding (moving the boundary), but an embedding $S^p \times S^q \to S^m$ formed by two such proper embeddings can be non-isotopic to the standard embedding.

Also note that any embedding symmetric w.r.t. $S^p \times S^{q-1}$ is isotopic to the standard embedding, but there are embeddings symmetric w.r.t. $S^{p-1} \times S^q$ and non-isotopic to the standard embedding.

4. The Whitehead Torus and the Whitehead almost concordance

Proof of the Extension Lemma 2.6.

Proof of (a). Add a strip to f_0, i.e. extend it to an embedding

$$f_0' : T^{0,q} \bigcup_{S^0 \times D^q_+ = \partial D^1_+ \times D^q_+} D^1_+ \times D^q_+ \to \mathbb{R}^s \quad \text{which is a smooth embedding on} \quad D^1_+ \times D^q_+.$$

This embedding contains connected sum of the components of f_0. The union of f_0' and the cone over the connected sum forms an embedding $D^1_+ \times S^q \to \mathbb{R}^{s+1}_+$. This latter embedding can clearly be shifted to a proper embedding $f_{1,+}$.

If f_0 is almost smooth, then $f_{1,+}$ can be made almost smooth. If f_0 is smooth, then the obstruction to smoothing the above almost smooth $f_{1,+}$ equals to the class in C^{s-q}_q of the connected sum of the components of f_0. So this obstruction is zero and we may assume that $f_{1,+}$ is smooth.

Given a CAT embedding $f_{p-1,+} : T^{p-1,q}_+ \to \mathbb{R}^{s+p-1}_+$ we rotate it in $\mathbb{R}^{s+p}_+$ with respect to $\mathbb{R}^{s+p-2} = \partial\mathbb{R}^{s+p-1}_+$ to obtain an embedding $f_{p,+} : T^{p,q}_+ \to \mathbb{R}^{s+p}_+$. Clearly, $f_{p,+}$ is a CAT embedding for CAT=PL, AD, DIFF.

So define inductively $f_{p,+}$ for $p \geq 2$ (but not for $p = 1$!) starting with $f_{1,+} = f_+$. Then set $f_p = \partial f_{p+1,+}$. Clearly, f_p is mirror-symmetric for $p \geq 1$. $\square$

Proof of (b). Let W_0 be a (almost) concordance between f_0 and g_0. We have

$$\mathbb{R}^{s+1}_+ \cup \mathbb{R}^s \times I \cup \mathbb{R}^{s+1}_- \cong \mathbb{R}^{s+1} \quad \text{and} \quad T^{1,q}_+ \cup T^{0,q} \times I \cup T^{1,q}_- \cong T^{1,q}.$$

Let $F_0 : T^{1,q} \to \mathbb{R}^{s+1}$ be an (almost) embedding obtained from $f_{1+} \cup g_{1+} \cup W_0$ by these homeomorphisms.

Now analogously to the proof of (a) extend F_0 to an (almost) embedding

$$T^{1,q} \bigcup_{S^1 \times D^q_+ = \partial D^2_+ \times D^q_+} D^2_+ \times D^q_+ \to \mathbb{R}^{s+1} \quad \text{which is a smooth embedding on} \quad D^2_+ \times D^q_+$$

and then to a map $F_+ : T^{2,q}_+ \to \mathbb{R}^{s+2}_+$. If F_0 is an embedding, then F_+ is also an embedding. If F_0 is an almost embedding, then $\Sigma(F_0) \cap x \times S^{q+1} = \emptyset$. Hence by the above construction $\Sigma(F_+) \cap x \times S^{q+1} = \emptyset$, i.e. F_+ is also an almost embedding.

This (almost) embedding F_+ can clearly be shifted to a proper (almost) concordance W_{1+} between f_{1+} and g_{1+}. If F_0 is almost smooth, then W_{1+} can be made almost smooth.

If W_0 is a smooth concordance, then the complete obstruction to smoothing W_{1+} is in C_{q+1}^{s-q} [BH70, Bo71]. If we change a smooth concordance W_0 between f_0 and g_0 by connected sum with a smooth sphere $h : S^{q+1} \to \mathbb{R}^s \times I$, then W_{1+} changes by a connected sum with the cone over s. Hence the obstruction to smoothing W_{1+} changes by $[h] \in C_{q+1}^{s-q}$ [BH70, Bo71]. Therefore by changing W_0 inside its almost smooth isotopy class (and modulo $T^{1,q}$) we can make W_{1+} smooth. So we may assume that W_{1+} is smooth.

In this paragraph assume that f_p and g_p not only are mirror-symmetric but also are obtained from the restrictions $f_{1+}, g_{1+} : T_+^{1,q} \to \mathbb{R}_+^{s+1}$ of f_1 and g_1 as in the proof of the Extension Lemma 2.6.a. (This simpler case is sufficient to the proof of most parts of the Whitehead Torus Theorem 2.5.) In this case we can define W_p from W_{1+} analogously to the 'rotation' construction of f_p from f_{1+} in the proof of the Extension Lemma 2.6.a.

The general case is proved as follows. Define W_{1-} by symmetry to W_{1+}. The two maps W_{1+} and W_{1-} fit together to give the required concordance W_1 between f_1 and g_1 (in the smooth category we can indeed make W_1 smooth). So we define W_p inductively using the following assertion.

If $f_p, g_p : T^{p,q} \to \mathbb{R}^m$ are mirror-symmetric embeddings whose restrictions $f_{p-1}, g_{p-1} : T^{p-1,q} \to \mathbb{R}^{m-1}$ are (almost) concordant and $m \geq 2p + q + 2$, then f_p and g_p are mirror-symmetrically (almost) concordant.

This assertion was proved in the first four paragraphs of this proof for $p = 1$, and the proof for arbitrary p is analogous. $\square$

Note that f_+ can be constructed from f_0 not explicitly but by the Haefliger-Irwin-Zeeman Embedding Theorem, cf. [Hu63]. Using the corresponding Unknotting Theorem we obtain that for $p \geq 1$, $m \geq 2p+q+2$ and PL or AD category a map $\hat{\mu}'_p : KT_{p-1,q}^{m-1} \to KT_{p,q}^m$ is uniquely defined by the condition that $\hat{\mu}'_p(f)$ is a mirror-symmetric extension of f. Clearly, $\tau_p \hat{\mu}''_p = \hat{\mu}'_p \tau_{p-1}$, where the homomorphism $\hat{\mu}''_p$ is induced by 'adding one vector' map $V_{m-q-1,p} \to V_{m-q,p+1}$.

Proof of the non-triviality of β-invariant.

Construction of the Whitehead almost concordance Ω_p between ω_p and the standard embedding. We start with the case $p = 0$, for which we use the idea of [Ha62, §4]. Consider the half-space $\mathbb{R}_+^{3l+1}$ of coordinates

$$(x, y, z, t) = (x_1, \dots, x_l, y_1, \dots, y_l, z_1, \dots, z_l, t), \quad t \geq 0.$$

Take three $2l$-disks $D_x, D_y, D_z \subset \mathbb{R}_+^{3l+1}$ given by the equations

$$\begin{cases} x = 0 \\ y^2 + 2z^2 + t^2 = 1 \end{cases}, \qquad \begin{cases} y = 0 \\ z^2 + 2x^2 + 2t^2 = 1 \end{cases} \qquad \text{and} \quad \begin{cases} z = 0 \\ x^2 + 2y^2 + 1.5t^2 = 1 \end{cases},$$

respectively. These disks bound Borromean rings.

Join D_x and D_y by a half-tube $D_+^{2l-1} \times I$ such that $\partial D_+^{2l-1} \times I \subset \mathbb{R}^{3l}$. We obtain a self-intersecting disk bounding the second component of the Whitehead link. The union of this disk and D_z form a smooth map $\Omega'_0 : S^0 \times D^{2l} \to \mathbb{R}_+^{3l+1}$ whose restriction to the boundary is ω_0.

On the intersection $D_x \cap D_z$ we have $y^2 + t^2 = 2y^2 + 1.5t^2 = 1$, hence $D_x \cap D_z = \emptyset$. Analogously $D_y \cap D_z = \emptyset$. So $\Sigma(\Omega'_0) \subset D_x \cap D_y$ and hence Ω'_0 is an almost embedding. Deleting from $\mathbb{R}_+^{3l+1}$ a small regular neighborhood of an arc joining a point from D_z to a point from $D_x - D_y$ we obtain an almost concordance Ω_0 between the Whitehead link and the standard link.

For the smooth category we modify the above Ω_0 by an almost smooth isotopy to obtain a smooth almost concordance Ω_0 between $\omega_{0,DIFF}$ and the standard link.

The Whitehead almost concordances Ω_p are constructed inductively by the Extension Lemma 2.6.b starting from Ω_0. $\square$

For the proof of the non-triviality (and thus of the Whitehead Torus Example 1.7) calculation of $\beta(\Omega_0)$ below can be omitted, because we only need that $\beta(\Omega_0) = \beta(\Omega_1) \neq 0$, but not that $\beta(\Omega_0) = 1$. We prove $\beta(\Omega_0) \neq 0$ simpler than $\beta(\Omega_0) = 1$ as follows: if $\beta(\Omega_0) = 0$, then by the completeness of β-invariant ω_0 is concordant to the standard link, which contradicts to $\lambda_{21}(\omega_0) \neq 0$.

Proof that $\beta(\Omega_p) = 1$. First we prove the case $p = 0$. The set $D_x \cap D_y$ is an $(l-1)$-sphere in $\mathbb{R}^{3l+1}_+$ given by the equations

$$\begin{cases} x = y = 0 \\ z^2 = t^2 = 1/3 \end{cases}.$$

Clearly, $[\Sigma(\Omega_0)]$ is represented by $\Omega_0^{-1}(D_x \cap D_y)$. The sphere $D_x \cap D_y$ bounds in D_x and in D_y two l-disks C_x and C_y given by the equations

$$\begin{cases} x = y = 0 \\ z^2 \le 1/3 \\ 2z^2 + t^2 = 1 \end{cases} \quad \text{and} \quad \begin{cases} x = y = 0 \\ z^2 \le 1/3 \\ z^2 + 2t^2 = 1 \end{cases}.$$

Then $\Omega_0 C$ is represented by the l-sphere $C_x \cup C_y$. This sphere bounds in $\mathbb{R}^{3l+1}$ an $(l+1)$-disk D given by equations

$$\begin{cases} x = y = 0 \\ z^2 \le 1/3 \\ \frac{1-z^2}{2} \le t^2 \le 1 - 2z^2 \end{cases}.$$

By the Alexander-Pontryagin duality $\beta(\Omega_0)$ is the algebraic sum of the intersection points of $D \cap D_z$. It is easy to check that $D \cap D_z = \{(0,0,0,\sqrt{2/3})\}$. Thus $\beta(\Omega_0) = \pm 1$, and we can choose orientations so that $\beta(\Omega_0) = 1$ for the almost smooth Ω_0. Since adding an isotopy to an almost concordance does not change β-invariant, this holds also for the smooth Ω_0.

We have $\beta(\Omega_p) = \beta(\Omega_0) = 1$ by the part (b) of the following Proposition. $\square$

Proposition. *(a) If $W : T^{p,2l} \to \mathbb{R}^{3l+p+1}$ is a mirror-symmetric almost embedding and $W_0 : T^{p-1,2l} \to \mathbb{R}^{3l+p}$ is the restriction of W, then $\beta(W) = \beta(W_0)$.*

(b) If $W : T^{p,2l-1} \times I \to \mathbb{R}^{3l+p} \times I$ is a mirror-symmetric almost concordance between embeddings and $W_0 : T^{p-1,2l-1} \times I \to \mathbb{R}^{3l+p-1} \times I$ is the restriction of W, then $\beta(W) = \beta(W_0)$.

Proof. First we prove (a). As in the definition of β we take chains

$$C_0 \quad \text{and} \quad C_\pm \quad \text{so that} \quad \partial C_0 = [\Sigma(W_0)], \quad \partial C_\pm = [\Sigma(W_\pm)] \pm C_0 \quad \text{and} \quad C_\pm \cap x \times S^{2l-1} = \emptyset.$$

Then $\partial(C_+ + C_-) = [\Sigma(W)]$. Let $D \subset \mathbb{R}^{3l+p}$ be a $(2l+1)$-chain represented by a ball whose boundary is $W(x \times S^{2l}) = W_0(x \times S^{2l}) \subset \mathbb{R}^{3l+p}$. If l is even, then the chain D is with integer coefficients and so the representing disk should be oriented. Then by the duality we have

$$\beta(W) = D \cap W(C_+ + C_-) = D \cap W_0(C_0) = \beta(W_0).$$

The proof of (b) is analogous. We only take as $D \subset \mathbb{R}^{3l+p-1} \times I$ a $(2l+1)$-ball whose boundary is the union of

$$W(x \times S^{2l-1} \times I) = W_0(x \times S^{2l-1} \times I) \subset \mathbb{R}^{m-1} \times I$$

and two $2l$-balls in $\mathbb{R}^{3l+p-1}\times\{0,1\}$ spanned by $W(x\times S^{2l-1}\times i)$, $i=0,1$. $\square$

Proof of the Whitehead Torus Theorem 2.5.

Invariance Lemma. *If $f,g:N\to\mathbb{R}^m$ are embeddings of a closed n-manifold N such that there is a codimension 0 ball $B\subset N$ and a homotopy $N\times I\to\mathbb{R}^m\times I$ between f and g whose self-intersection set is contained in $B\times I$, then $\alpha_\infty(f)=\alpha_\infty(g)$.*

We present a proof of this folklore result for completeness. Cf. [Sk02, Theorem 5.2.α, Sk05, Almost Embeddings Theorem (a)].

Proof. It suffices to prove that for each i there is an equivariant deformation retraction

$$R:\widetilde{N}^i\to\overline{N}^i:=\{(x_1,\dots,x_i)\in\widetilde{N}^i\mid \#(\{x_1,\dots,x_i\}\cap B)\le 1\}.$$

Take $a\in N$ and a metric on N such that B is the closed 1-neighborhood of a and the close 2-neighborhood B_2 of a is a ball. Identify B_2 with the ball of radius 2 in $\mathbb{R}^n$. Now, given $(x_1,\dots,x_i)\in\widetilde{N}^i-\overline{N}^i$, there exist $u\ne v$ such that $|a-x_u|$ and $|a-x_v|$ are the smallest among all $|a-x_s|$ (possibly $|a-x_u|=|a-x_v|$). Then $x_u,x_v\in B$. Assume that $|a-x_u|\ge|a-x_v|$. Then $|a-x_u|\ne 0$. Take a piecewise-linear function

$$M:[0,2]\to[0,2]\quad\text{such that}\quad M(0)=0,\quad M(|a-x_u|)=1\quad\text{and}\quad M(2)=2.$$

Define a map

$$R:\widetilde{N}^i-\overline{N}^i\to\overline{N}^i\quad\text{by}\quad R_s(x_1,\dots,x_i):=\begin{cases}x_sM(|a-x_s|), & x_s\in B_2\\ x_s & x_s\notin B_2\end{cases}.$$

Extend this map identically over $\widetilde{N}^i$. We obtain the required equivariant deformation retraction. $\square$

Proof of the α- and the α_∞-triviality. By the non-triviality of β-invariant there is an almost concordance Ω_p between ω_p and the standard embedding (only the existence of Ω_p is necessary here). We may assume that in fact Ω_p is an *almost isotopy* (either checking that the construction could give in fact an almost isotopy, cf. sketch of construction of Ω_0 in §2, or applying the Melikhov 'almost concordance implies almost isotopy in codimension at least 3' theorem). Since $2l+p+p+1<3l+p+1$, by general position we may assume that $\Sigma(\Omega_p)\cap S^p\times I=\emptyset$. Therefore the α- and the α_∞-triviality follow by the Invariance Lemma. $\square$

An alternative proof of the α-triviality (without use of the Invariance Lemma) can be obtained observing that α changes as a suspension under the construction of the Extension Lemma 2.6.a (cf. [Sk02, Decomposition Lemma 7.1, Ke59, Lemma 5.1]) and proving that $\alpha(\omega_0)=0$ (either because $\alpha(\omega_0)=\pm\Sigma\lambda(\omega_0)$ or because ω_0 is almost concordant to the standard link).

Proof of the non-triviality. Suppose to the contrary that there is an almost smooth isotopy F between ω_p and the standard embedding. Then by the obstruction, the independence and the non-triviality of β-invariant we have $0=\beta(F)=\beta(\Omega_p)\ne 0$, which is a contradiction. $\square$

Proof of the Relation Theorem 2.7.

Symmetry Lemma. *For*

$$1\le p\le l-2\quad\textit{or}\quad 1=p=l-1\quad\textit{we have}\quad \omega_p=\sigma_p\omega_p=\sigma_{3l+p}\omega_p=-\sigma_{2l-1}\omega_p$$

in the smooth category, except the first equality for l even, which holds only in the almost smooth category.

Proof. By construction ω_p is mirror symmetric for $p > 0$, i.e. $\sigma_p\omega_p = \sigma_{3l+p}\omega_p$. By definition of the inverse element $\sigma_{3l+p}\omega_p = -\sigma_{2l-1}\omega_p$.

Recall that ω_p and $\sigma_p\omega_p$ are mirror-symmetric extensions of ω_{p-1} and of $\sigma_{p-1}\omega_{p-1}$. Hence analogously to the Extension Lemma 2.6.b (by the assertion at the end of its proof) it suffices to prove that there exists a concordance (not necessarily mirror-symmetric) between ω_1 and $\sigma_1\omega_1$.

By the non-triviality of β-invariant Ω_1 and $\sigma_1\Omega_1$ are smooth almost concordances from ω_1 and from $\sigma_1\omega_1$ to the standard embedding. Since σ_1 reverses the orientation of $S^1 \times S^{2l-1} \times I$ but not of $x \times S^{2l-1} \times I$ and not of $\mathbb{R}^{3l+1} \times I$, it follows that σ_1 preserves the orientation of chains $[\Sigma(\Omega_1)]$, C and $\Omega_1 C$ from the definition of β. Therefore $\beta(\sigma_1\Omega_1) = \beta(\Omega_1)$, so $\beta(\sigma_1\Omega_1 \cup \overline{\Omega}_1) = 0$. Hence by the completeness of β-invariant $\sigma_1\omega_1$ is almost smoothly concordant to ω_1.

If $l \geq 3$ is odd, then the complete obstruction to smoothing this almost smooth concordance is in $C_{2l}^{l+1} = 0$ [Ha66', Mi72], so $\sigma_1\omega_1$ is smoothly concordant to ω_1. $\square$

The relation $\omega_p = \sigma_p\omega_p$ (and the τ-relation) in the smooth category holds also for $p \geq 3$ and $l \equiv \pm 2 \ mod 12$ by the above (and below) proof and $C_{2l+2}^{l+1} = 0$ [Mi72, Corollary G].

Note that $\omega_{0,1} = \sigma_0\omega_{0,1}$ but $\omega_{0,l} \neq \sigma_0\omega_{0,l}$ for $l \notin \{1, 3, 7\}$ by the Haefliger Theorem below. It would be interesting to know if $\omega_{0,l} = \sigma_0\omega_{0,l}$ for $l \in \{3, 7\}$.

The Haefliger Theorem. [Ha62', §3] *For $l \notin \{1, 3, 7\}$ two smooth links $f, g : T^{0,2l-1} \to \mathbb{R}^{3l}$ are*

PL isotopic if and only if $\lambda_{12}(f) = \lambda_{12}(g)$ and $\lambda_{21}(f) = \lambda_{21}(g)$

smoothly isotopic if and only if they are PL isotopic and their restrictions to components are smoothly isotopic.

Definition of the linking coefficients λ_{ij}. Suppose that $p, q \leq m-3$ and $f : S^p \sqcup S^q \to S^m$ is an embedding. Take orientations of S^p, S^q, S^m and D^{m-p}. Take an embedding $g : D^{m-p} \to S^m$ such that gD^{m-p} intersects fS^p transversely at exactly one point of sign $+1$. Then $g|_{\partial D^{m-p}} : \partial D^{m-p} \to S^m - fS^p$ is a homotopy equivalence. Let $h_f : S^m - fS^p \to \partial D^{m-p}$ be a homotopy inverse. Clearly, the homotopy class of h_f does not depend on the choice of g. The linking coefficient is

$$\lambda_{12}(f) \ = \ [f|_{S^q} : S^q \to S^m - fS^p \xrightarrow{h_f} \partial D^{m-p}] \ \in \ \pi_q(S^{m-p-1}).$$

Analogously is defined $\lambda_{21}(f) \in \pi_p(S^{m-q-1})$.

For odd $l \notin \{1, 3, 7\}$ we have $2\omega_0 = 0$ by the Haefliger Theorem because $2[\iota_l, \iota_l] = 0$. The following proof works for *each* odd $l \geq 3$ and, for the almost smooth category, is independent on the Haefliger Theorem.

Proof of the 2-relation. By the Extension Lemma 2.6.b it suffices to prove that $2\omega_0 = 0$. Take a smooth almost concordance Ω_0 given by the non-triviality of β-invariant. Then $\sigma_{3l}\sigma_{2l-1}\Omega_0 \cup \overline{\Omega}_0$ is a smooth almost concordance between $\sigma_{3l}\sigma_{2l-1}\omega_0 = -\omega_0$ and ω_0. Since l is odd, we have

$$\beta(\sigma_{3l}\sigma_{2l-1}\Omega_0 \cup \overline{\Omega}_0) = \pm\beta(\Omega_0) - \beta(\Omega_0) = 0 \in \mathbb{Z}_2.$$

Therefore by the completeness of β-invariant ω_0 and $-\omega_0$ are almost smoothly concordant, and hence PL isotopic.

In the smooth category the restrictions of both ω_0 and $-\omega_0$ to their components are smoothly isotopic because $2\varphi = 0 \in C^{l+1}_{2l-1} \cong \mathbb{Z}_2$. Hence ω_0 and $-\omega_0$ are smoothly isotopic by the Haefliger Theorem. $\square$

Proof of the τ-relation. For $l \in \{3,7\}$ we have $\tau(w_{l,p}) = 0$ because $[\iota_l, \iota_l] = 0$. So the τ-relation follows from the 2-relation.

For $l \notin \{1,3,7\}$ let us prove that

(a) $\tau_0(w_{l,0}) = \omega_0 + \sigma_0\omega_0$ in the almost smooth category.

Recall that $\sigma_0\psi$ is the link obtained from a link $\psi : T^{0,q} \to \mathbb{R}^m$ by interchanging the components. Recall that

$$\lambda_{12}(\omega_0) = [\iota_l, \iota_l] \quad \text{and} \quad \lambda_{21}(\omega_0) = 0, \quad \text{hence} \quad \lambda_{12}(\sigma_0\omega_0) = 0 \quad \text{and} \quad \lambda_{21}(\sigma_0\omega_0) = [\iota_l, \iota_l].$$

Since linking coefficients are additive under connected sum, it follows that both linking coefficients of $\omega_0 + \sigma_0\omega_0$ are $[\iota_l, \iota_l]$.

Identify $\pi_{2l-1}(S^l)$ and $\pi_{2l-1}(V_{l+1,1})$ by the isomorphism μ_0'' (which we thus omit from the notation). Clearly, for each $\varphi \in \pi_{2l-1}(S^l)$ we have

$$\lambda_{12}(\tau\varphi) = \varphi \quad \text{and} \quad \lambda_{21}(\tau\varphi) = \lambda_{12}(\sigma_0\tau\varphi) = \lambda_{12}(\tau(a_l \circ \varphi)).$$

Here a_l is the antipodal map of S^l. The last equality follows because $\sigma_0\tau\varphi = \tau(a_l \circ \varphi)$. If l is odd, then $a_l \circ \varphi = \varphi$. If l is even, then $a_l \circ \varphi = -\varphi + [\iota_l, \iota_l] \circ h_0(\varphi)$ [Po85, complement to Lecture 6, (10)]. Since $h_0[\iota_l, \iota_l] = 2$ [Po85, Lecture 6, (7)], it follows that $a_l \circ [\iota_l, \iota_l] = [\iota_l, \iota_l]$. Therefore both linking coefficients of $\tau[\iota_l, \iota_l]$ are $[\iota_l, \iota_l]$.

So by the Haefliger Theorem $\tau_0(w_{l,0})$ is smoothly isotopic to $\omega_0 + \sigma_0\omega_0$ because the restrictions of both links to components are standard.

Recall that ω_p, $\sigma_p\omega_p$ and $\tau_p(w_{l,p})$ are mirror-symmetric extensions of ω_{p-1}, $\sigma_{p-1}\omega_{p-1}$ and $\tau_{p-1}(w_{l,p-1})$, respectively. Hence by the Extension Lemma 2.6.b $\tau_p(w_{l,p}) = \omega_p + \sigma_p\omega_p$ in the almost smooth category.

Recall that for l odd the complete obstruction to smoothing an almost smooth concordance $T^{1,2l-1} \times I \to \mathbb{R}^{3l+1} \times I$ is in $C^{l+1}_{2l} = 0$ [Ha66', Mi72]. Hence $\tau_1(w_{l,1})$ is smoothly concordant to $\omega_1 + \sigma_1\omega_1$ (the concordance is not necessarily mirror-symmetric). Hence analogously to the Extension Lemma 2.6.b $\tau_p(w_{l,p}) = \omega_p + \sigma_p\omega_p$ in the smooth category for $p \geq 1$.

Now the τ-relation follows by the relation $\sigma_p\omega_p = \omega_p$ of the Symmetry Lemma. $\square$

Note that the smooth version of (a) is false (because the smooth ω_0 is constructed by connected summation with φ).

5. Proof of the Torus Theorem 2.8

Denote $KT^m_{p,q,+,CAT} := \mathrm{Emb}^m_{CAT}(D^p_+ \times S^q)$. For $m \geq \max\{2p+q+2, q+3\}$ a group structure on $KT^m_{p,q,+}$ is defined analogously to that on $KT^m_{p,q}$ above. For the smooth category the sum is connected sum of q-spheres together with fields of p normal vectors.

Lemma 5.1. *For $p \leq q$, $m \geq p+q+3$ and $m \geq 2p+q+2$ there are homomorphisms*

$$\pi_q(V_{m-q,p+1}) \xrightarrow[\tau_+]{} KT^m_{p+1,q,+} \xrightarrow[\alpha]{} \pi^{m-1}_{eq}(\widetilde{T^{p+1,q}_+}) \xrightarrow[r]{} \pi^{m-1}_{eq}(\widetilde{T^{p,q}}) \xleftarrow[\sigma']{} \pi_q(V^{eq}_{m-q,p+1}).$$

Here r is the isomorphism induced by restriction.

The map αr is an isomorphism for $2m \geq 3q+2p+4$ in the smooth category and for $2m \geq 3q+2p+5$ in the PL category.

The map τ_+ is defined as follows (analogously to τ in §2). Represent $\varphi \in \pi_q(V_{m-q,p+1})$ as a mapping $D^p \times S^q \to D^{m-q}$. Define $\tau_+(\varphi)$ to be the composition $D^p \times S^q \overset{\varphi\times\mathrm{pr}_2}{\to} D^{m-q} \times S^q \subset \mathbb{R}^m$.

The map τ_+ is an isomorphism for $2m \ge 3q+4$ in the smooth category and for $2m \ge 3q+2p+5$ in the PL category.

The equivariant Stiefel manifold V^{eq}_{mn} *is the space of maps $S^{n-1} \to S^{m-1}$ which are equivariant with respect to the antipodal involutions. The map σ' is defined (as $\gamma^{-1}i_2\sigma$) and for $2m \ge 3q+p+4$ is proved to be an isomorphism in [Sk02, Proof of Torus Lemma 6.1, Sk07, Proof of Torus Lemma 6.1].*

Proof. Analogously to [Sk02, Torus Lemma 6.1, Sk07, Torus Lemma 6.1] there is an equivariant deformation retraction

$$\widetilde{T^{p+1,q}_+} \to \mathrm{adiag}(\partial D^{p+1}) \times S^q \times S^q \bigcup_{\mathrm{adiag}(\partial D^{p+1})\times \mathrm{adiag}\, S^q} D^{p+1} \times D^{p+1} \times \mathrm{adiag}\, S^q.$$

Hence by general position r is an isomorphism for $m \ge p+q+3$ and $m \ge 2p+q+2$.

Then the assertion on αr follows by [Ha63, 6.4, Sk02, Theorems 1.1.$\alpha\partial$ and 1.3.$\alpha\partial$].

The assertion on τ_+ in the PL category follows the assertion on αr and the assertion on $\alpha\tau$ of the Torus Theorem 2.8.

The assertion on τ_+ in the smooth category follows because for $2m \ge 3q+4$ every smooth embedding $S^q \to \mathbb{R}^m$ is smoothly isotopic to the standard one and by the (trivial) smooth version of the Fibering Lemma 7.1 below, cf. the Filled-Tori Theorem 7.3.a below. $\square$

Restriction Lemma 5.2. *For $p \ge 1$, $m \ge 2p+q+2$ and $2m \ge 3q+p+4$ there is the following commutative (up to sign) diagram with exact lines:*

$$\begin{array}{ccccccccc}
\xrightarrow[\lambda''_{q+1}]{} & \pi_q(S^{m-p-q-1}) & \xrightarrow[\mu'']{} & \pi_q(V_{m-q,p+1}) & \xrightarrow[\nu'']{} & \pi_q(V_{m-q,p}) & \xrightarrow[\lambda''_q]{} & \pi_{q-1}(S^{m-p-q-1}) \\
& \downarrow{\scriptstyle \Sigma^p} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau_+} & & \downarrow{\scriptstyle \Sigma^p} \\
\xrightarrow[\overline{\lambda}_{q+1}]{} & \pi_{p+q}(S^{m-q-1}) & \xrightarrow[\overline{\mu}]{} & \overline{KT}^m_{p,q} & \xrightarrow[\overline{\nu}]{} & KT^m_{p,q,+} & \xrightarrow[\overline{\lambda}_q]{} & \pi_{p+q-1}(S^{m-q-1}) \\
& \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \bar\alpha'} & & \downarrow{\scriptstyle \alpha' r} & & \downarrow{\scriptstyle =} \\
\xrightarrow[\lambda'_{q+1}]{} & \pi_{p+q}(S^{m-q-1}) & \xrightarrow[\mu']{} & \pi_q(V^{eq}_{m-q,p+1}) & \xrightarrow[\nu']{} & \pi_q(V^{eq}_{m-q,p}) & \xrightarrow[\lambda'_q]{} & \pi_{p+q-1}(S^{m-q-1})
\end{array}.$$

Here the the upper and the bottom lines are the exact sequences of the 'restriction' Serre fibrations $S^{m-p-q-1} \to V_{m-q,p+1} \to V_{m-q,p}$ and $\Omega_p S^{m-p-q-1} \to V^{eq}_{m-q,p+1} \to V^{eq}_{m-q,p}$. The homomorphism $\overline{\nu}$ is restriction-induced. The homomorphisms $\overline{\lambda}_q$ and $\overline{\mu}$ are defined below.

By $\bar\alpha'$ and $\alpha' r$ we denote the compositions of $\bar\alpha$ and αr with the isomorphism σ' of Lemma 5.1. Denote by $\rho = \rho_p : \pi_q(V_{m-q,p}) \to \pi_q(V^{eq}_{m-q,p})$ the inclusion-induced homomorphism. Then $\bar\alpha'\tau = \pm\rho_{p+1}$ and $\alpha' r\tau_+ = \pm\rho_p$.

For $p \ge 1$ and $2m = 3q+p+3 \ge 4p+2q+4$ all the diagram except $\bar\alpha'$ is still defined and commutative (up to sign), and the lines are exact.

Recall that $T^{p,q} = S^p \times S^q$ and $T^{p,q}_+ = D^p_+ \times S^q$. Denote

$$B^{p+q} := T^{p,q} - (\mathrm{Int}\, D^p_+ \times S^q_+ \cup \mathrm{Int}\, S^p \times D^q_+).$$

Take $x \in \operatorname{Int} D^p_-$. Denote by $G_0 : T^{p,q}_+ \to \mathbb{R}^m$ the standard embedding. Recall the definition of h_f from the definition of linking coefficients in §4.

Definition of $\overline{\lambda}_q$. Take an embedding $G : T^{p,q}_+ \to S^m$. Since $m \geq 2p+q+2$, by general position G has a unique up to isotopy extension to an embedding $G' : T^{p,q} - \operatorname{Int} B^{p+q} \to S^m$. Define $\overline{\lambda}_q(G)$ to be the homotopy class of the map

$$G'|_{\partial B^{p+q}} : \partial B^{p+q} \to S^m - G(x \times S^q) \overset{h_G}{\simeq} S^{m-q-1}.$$

Definition of $\overline{\mu}$. For $x \in \pi_{p+q}(S^{m-q-1})$ take a map $S^{p+q} \to S^m - G_0(T^{p+1,q}_+) \overset{h_{G_0}}{\simeq} S^{m-q-1}$ representing the class x. Define $\overline{\mu}(x)$ to be the connected sum of the embedding $G_0|_{T^{p,q}}$ with this map.

Proof of the exactness at $KT^m_{p,q,+}$ *in the Restriction Lemma 5.2.* By definition $\overline{\lambda}_q(G)$ is the obstruction to extending an embedding $G : T^{p,q}_+ \to S^m$ to an almost embedding $T^{p,q} \to \mathbb{R}^m$, so the sequence is exact at $KT^m_{p,q,+}$. $\square$

Proof of the exactness at $\overline{KT}^m_{p,q}$ *in the Restriction Lemma 5.2.* Clearly, $\overline{\nu\mu} = 0$. Let $G : T^{p,q} \to S^m$ be an almost embedding such that $\overline{\nu}(G) = 0$. Then G is isotopic to an almost embedding standard on $T^{p,q}_+$. Thus we may assume that G itself is standard on $T^{p,q}_+$. Since $m \geq 2p+q+2$, by general position G is isotopic relative to $T^{p,q}_+$ to an embedding standard outside B^{p+q}. Thus we may assume that G itself is standard outside B^{p+q}.

Hence $G|_{B^{p+q}}$ and $G_0|_{B^{p+q}}$ form together a map $S^{p+q} \to S^m - G_0(\operatorname{Int} T^{p,q}_+) \simeq S^{m-q-1}$. Let x be the homotopy class of this map. Then $G = \overline{\mu}(x)$ (because $[G(B^{p+q}) \cup G_0(B^{p+q})] \cap GT^{p,q}_+ = \emptyset$, so there is an isotopy of S^m rel $G(T^{p,q}_+)$ moving $G_0(T^{p+1,q}_+)$ to a neighborhood in S^m of $GT^{p,q}_+$ rel $G\partial T^{p,q}_+$ that misses $G(B^{p+q}) \cup G_0(B^{p+q})$).

Definition of $\lambda(f) \in \pi_{p+q}(S^{m-q-1})$ *for an almost embedding* $f : T^{p,q} \to S^m$ *coinciding with* G_0 *on* $T^{p,q}_+$. The restrictions of f and G_0 onto $T^{p,q}_-$ coincide on the boundary and so form together a map

$$\lambda'(f) : S^p \times S^q \to S^m - f(x \times S^q) \simeq S^{m-q-1}.$$

The map $S^p \times S^q \to (S^p \times S^q)/(S^p \vee S^q) \cong S^{p+q}$ induces a 1–1 correspondence between homotopy classes of maps $S^p \times S^q \to S^{m-q-1}$ and $S^{p+q} \to S^{m-q-1}$. Let $\lambda(f)$ be the homotopy class of the map corresponding to $\lambda'(f)$. (Note that $x = \lambda(G)$ in the previous proof.)

Proof of the exactness at $\pi_{p+q}(S^{m-q-1})$ *in the Restriction Lemma 5.2.* First we prove that $\overline{\mu}\overline{\lambda}_{q+1} = 0$ (note that only this part of the exactness is required for the proof of the main results). Take an embedding $\psi : T^{p,q+1}_+ \to \mathbb{R}^{m+1}$. Represent $S^{q+1} = D^{q+1}_< \cup D^q \times I \cup D^{q+1}_>$. Analogously to the Standardization Lemma 2.1 for $m \geq 2p+q+2$ in the PL category and for $m \geq 2p+q+3$ in the smooth category making an isotopy we can assume that

$\psi|_{D^p_+ \times D^{q+1}_>}$ is the standard proper embedding into $\mathbb{R}^m \times [1, +\infty)$,

$\psi|_{D^p_+ \times D^{q+1}_<}$ is the standard proper embedding into $\mathbb{R}^m \times (-\infty, 0]$, and

$\psi|_{D^p_+ \times D^{q+1} \times I}$ is a concordance between standard embeddings.

The latter concordance is ambient, so there is a homeomorphism

$$\Psi : \mathbb{R}^m \times I \to \mathbb{R}^m \times I \quad \text{such that} \quad \Psi(\psi(y,0),t) = \psi(y,t) \quad \text{for each } y \in D^p_+ \times D^{q+1},\ t \in I.$$

So the standard embedding $G_0 : T^{p,q} \to \mathbb{R}^m$ is concordant to an embedding $\overline{\psi}_1 : T^{p,q} \to \mathbb{R}^m \times 1$ defined by $\overline{\psi}_1(y) := \Psi(G_0(y), 1)$. The concordance $\Psi|_{G_0(T^{p,q}) \times I}$ together with the

standard extensions of $\psi|_{D^p_+\times D^{q+1}_{>,<}}$ to $S^p\times D^{q+1}_{>,<}$ form an extension of ψ to $T^{p,q+1}-B^{p+q+1}$. Hence $\lambda(\overline{\psi}_1)=\overline{\lambda}_{q+1}(\psi)=\lambda(\overline{\mu}\overline{\lambda}_{q+1}(\psi))$. Clearly, $\lambda(\overline{\mu}x)=x$, and

almost embeddings $f,g:T^{p,q}\to S^m$ coinciding with G_0 on $T^{p,q}_+$ are almost concordant relative to $T^{p,q}_+$ if and only if $\lambda(f)=\lambda(g)$.

Therefore $\overline{\mu}\overline{\lambda}_{q+1}(\psi)=\overline{\psi}_1$ is concordant to the standard embedding.

Now we prove that $\ker\overline{\mu}\subset\operatorname{im}\overline{\lambda}_{q+1}$. If $\overline{\mu}(x)=0$, then take the restriction $T^{p,q}_+\times I\to\mathbb{R}^m\times I$ of an almost concordance between $\overline{\mu}(x)$ and the standard embedding. This restriction is a concordance between standard embeddings and so can be completed to an embedding $\psi:T^{p,q+1}_+\to\mathbb{R}^{m+1}$. Analogously to the above it is proved that $\overline{\lambda}_{q+1}(\psi)=x$. $\square$

For the sequel we need $\Pi^{m-1}_{p,q}:=\pi^{m-1}_{eq}(S^p\times S^{2q})$, where the involution on $S^p\times S^{2q}$ is the product of the antipodal involution on S^p and the symmetry with respect to $S^q\subset S^{2q}$. The group structure on Π^{m-1}_{pq} is defined as S^p-parametric version of the group structure on $[S^{2q},S^{m-1}]$, see the details in [Sk02, Torus Lemma 6.1, Sk07, Torus Lemma 6.1]. An element $\varphi\in\pi_q(V^{eq}_{m-q,p+1})$ can be considered as a map $\varphi:S^p\times S^q\to S^{m-q-1}$ such that $\varphi(-x,y)=-\varphi(x,y)$ for each $x\in S^p$. Define a homomorphism $\sigma:\pi_q(V^{eq}_{m-q,p+1})\to\Pi^{m-1}_{p,q}$ by setting $\sigma(\varphi)$ to be the q-fold S^p-fiberwise suspension of such a map φ, i.e. $\sigma(\varphi)|_{x\times S^{2q}}=\Sigma^q(\varphi|_{x\times S^q})$. By [Sk02, Proof of Torus Lemma 6.1, Sk07, Proof of Torus Lemma 6.1], σ is an isomomorphism for $2m\ge 3p+q+4$.

Proof of the commutativity in the Restriction Lemma 5.2. In this proof 'commutativity' means 'commutativity up to sign'. The commutativity of the middle squares is trivial.

It is clear and well-known [HH62] that the big vertical rectangulars are commutative (because the composition $\pi_q(S^{m-p-q-1})\overset{\Sigma^p}{\to}\pi_{p+q}(S^{m-q-1})\cong\pi_q(\Omega_pS^{m-q-1})$ is induced by ρ_{p+1}).

Let us prove the commutativity of the left upper square. The case $p=0$ is clear. The case of arbitrary p follows by the Extension Lemma 2.6.b because $m\ge 2p+q+2$ and $\tau_p\mu''_px:T^{p,q}\to\mathbb{R}^m$ is a mirror-symmetric extension of $\tau_{p-1}\mu''_{p-1}x:T^{p-1,q}\to\mathbb{R}^{m-1}$ and $\overline{\mu}_p\Sigma^px:T^{p,q}\to\mathbb{R}^m$ is a mirror-symmetric extension of $\overline{\mu}_{p-1}\Sigma^{p-1}x:T^{p-1,q}\to\mathbb{R}^{m-1}$.

The commutativity of the right upper square for $2m\ge 3q+p+3$ follows by the commutativity of the right bottom square (proved below) and the commutativity of a right big vertical rectangular.

By [Sk02, Torus Lemma 6.1] the bottom line can be identified with the following sequence:

$$\xrightarrow[\lambda_{q+1}]{}\pi_{p+2q}(S^{m-1})\xrightarrow[\mu]{}\Pi^{m-1}_{p,q}\xrightarrow[\nu]{}\Pi^{m-1}_{p-1,q}\xrightarrow[\lambda_q]{}\pi_{p+2q-1}(S^{m-1}).$$

Under this identification the vertical arrows marked with the equality are identified with the suspension homomorphisms Σ^q (which are isomorphisms because $2m\ge 3q+p+3$). Hence the commutativity of the left bottom square follows by the commutativity of the left bottom square from [Sk02, Decomposition Lemma 7.1].

Under this identification the map λ'_q is identified with the map λ_q defined as the obstruction to extension over $T^{p,2q}_+$. Hence the commutativity of the right bottom square for $2m\ge 3q+p+3$ follows from the relation $\lambda_q\alpha'r=\Sigma^q\overline{\lambda}_q$ which is clear by the proof of [Sk02, proof of Torus Lemma 6.1, Sk07, proof of Torus Lemma 6.1]. $\square$

Proof of the relations $\bar{\alpha}'\tau=\pm\rho_{p+1}$ and $\alpha'r\tau_+=\pm\rho_p$ in the Restriction Lemma 5.2. It suffices to prove the first relation. It reduces to $\alpha\tau=\pm\sigma\rho_{p+1}$. Represent $\varphi\in\pi_q(V_{m-q,p+1})$ as a map $S^p\times S^q\to S^{m-q-1}$. The relation is proved for $p=0$ using the representation $S^{m-1}\cong S^{m-q-1}*S^{q-1}$ and deforming $\alpha\tau(\varphi)$ to the (S^p-fiberwise) suspension $\sigma(\varphi)$ of φ

[Ke59]. The same relation for $p>0$ is proved by applying this deformation for each $x\in S^p$ independently. $\square$

Proof of the assertion on $\bar\alpha$ in the smooth category in the Torus Lemma 2.7. Consider the bottom two lines of the diagram from the Restriction Lemma 5.2. By Lemma 5.1 and [Ha63, 6.4, Sk02, Theorem 1.1.$\alpha\partial$] the map $\alpha' r$ is an isomorphism for $2m\ge 3q+2p+2$ and an epimorphism for $2m=3q+2p+1$. Hence by the 5-lemma it follows that $\bar\alpha$ is an isomorphism for $2m\ge 3q+2p+2$. $\square$

Proof of the assertions on $\alpha\tau$ and the 'moreover' part in the Torus Theorem 2.8. Let $n=p+1$ and $l=m-p-q$. Identify the sets of Lemma 5.1 by the isomorphisms of Lemma 5.1.

In the following two paragraphs we reproduce the argument from [HH62, (1.1)]. The proof is by induction on n. For $n=1$ the map ρ_n is an isomorphism because $V_{l+1,1}\cong V^{eq}_{l+1,1}\cong S^l$. Suppose now that $n\ge 2$. Consider the following diagram formed by the upper and the bottom line of the diagram from the Restriction Lemma 5.2:

$$\begin{array}{ccccccccc}
\pi_{q+1}(V_{l+n,n-1}) & \to & \pi_q(S^l) & \xrightarrow[\mu'']{} & \pi_q(V_{l+n,n}) & \xrightarrow[\nu'']{} & \pi_q(V_{l+n,n-1}) & \xrightarrow[\lambda'']{} & \pi_{q-1}(S^l)\\
\downarrow{\scriptstyle\rho_{n-1}} & & \downarrow{\scriptstyle\Sigma^{n-1}} & & \downarrow{\scriptstyle\rho_n} & & \downarrow{\scriptstyle\rho_{n-1}} & & \downarrow{\scriptstyle\Sigma^{n-1}}\\
\pi_{q+1}(V^{eq}_{l+n,n-1}) & \to & \pi_{q+n-1}(S^{l+n-1}) & \xrightarrow[\mu']{} & \pi_q(V^{eq}_{l+n,n}) & \xrightarrow[\nu']{} & \pi_q(V^{eq}_{l+n,n-1}) & \xrightarrow[\lambda']{} & \pi_{q+n-2}(S^{l+n-1})
\end{array}.$$

Suppose that $q\ge 1$ (the argument for $q=0$ is the same, only the right-hand terms in the above diagram should be replaced by zeros, since both restrictions inducing ν'' and ν' are surjective).

Let us prove that ρ_n is an isomorphism for $2m\ge 3q+2p+4\Leftrightarrow q\le 2l-2$. By the Freudenthal Suspension Theorem Σ^{n-1} are isomorphisms. By the inductive hypothesis ρ_{n-1} are isomorphisms. So by the 5-lemma ρ_n is an isomorphism.

Let us prove the 'moreover' part. We have $q=2l-1$ and $n\ge 2$. Hence by the Freudenthal Suspension Theorem the right-hand Σ^{n-1} is an isomorphism and the left-hand Σ^{n-1} is an epimorphism whose kernel is generated by $[\iota_l,\iota_l]$. By the inductive hypothesis ρ_{n-1} are isomorphisms. Therefore if we factorize $[\iota_l,\iota_l]\in\pi_q(S^l)$ and $w_{l,n-1}\in\pi_q(V_{l+n,n})$, by the 5-lemma we obtain that ρ_n an epimorphism whose kernel is generated by $w_{l,n-1}$. $\square$

Remarks. (a) *An alternative proof that the map τ of the Torus Theorem 2.8 is an isomorphism for $m\ge\max\{2p+q+2,3q/2+p+2\}$ in the PL category and for $m\ge\max\{2p+q+3,3(q+p)/2+2\}$ in the smooth category.* Recall that the forgetful map $KT^m_{p,q}\to\overline{KT}^m_{p,q}$ is an isomorphism by [Sk02, Theorem 2.2.αq] and general position (and, in the smooth category, by smoothing). Consider the upper two lines of the diagram from the Restriction Lemma 5.2. The map τ_+ is an isomorphisms for $2m\ge 3q+2p+5$ by the assertion on τ_+ of Lemma 5.1, and is an epimorphism for $2m=3q+2p+4$. So by the induction on q using the 5-lemma we obtain that τ is an isomorphism.

(b) *An alternative proof of the commutativity of the upper right square from the Restriction Lemma 5.2 for $p=1$ and $q\le 2(m-q-1)-2$.* We omit index q. Recall that $\lambda''\Sigma x=(1-(-1)^{m-q})x$ for $x\in\pi_{q-1}(S^{m-q-2})$ [JW54] (in that paper λ'' was denoted by Δ). On the other hand, $\bar\lambda y$ is the signed sum of linking coefficients of the link $y|_{T^{0,q}}$, thus $\bar\lambda y=(1-(-1)^{m-q})y$. By the Freudenthal Suspension Theorem it follows that for each $y\in\pi_q(S^{m-q-1})$ there is $x\in\pi_{q-1}(S^{m-q-2})$ such that $y=\Sigma x$. Hence $\Sigma\lambda''y=\Sigma\lambda''\Sigma x=\Sigma(1-(-1)^{m-q})x=\bar\lambda y$.

(c) *An alternative proof of the commutativity of the upper left square from the Restriction Lemma 5.2 for $2m\ge 3q+2p+2$.* Follows by the commutativity of the left bottom square

and the commutativity of a left square of a diagram from the proof of the assertions on $\alpha\tau$, because $\bar\alpha$ is injective for $2m \ge 3q+2p+2$ (which is proved using only the two bottom lines of the diagram).

(d) *The map $\alpha\tau$ is an epimorphism for $q=2l$, $m=3l+p+1$, $p\ge 2$ and $l+1$ not a power of 2.* (The condition that $l+1$ is not a power of 2 can be replaced by $w_{l,p}\ne 0$.) This can be proved using the diagram from the proof of the assertions on $\alpha\tau$ and the diagram choice as if in the proof of the Triviality Lemma 6.2. This fact together with the assertion on $\bar\alpha$ of the Torus Theorem 2.8 for the smooth category implies the Triviality Lemma 6.2 below.

6. Proof of the properties of β-invariant

Proof of the completeness in the β-invariant Theorem 2.9. By general position, since $3l+p+1\ge 2(p+1)+(2l-1)+1$, there is a homotopy of F rel $T^{p,2l-1}_+\times I\cup T^{p,2l-1}\times\{0,1\}$ to an almost concordance (denoted by the same letter F) for which $\Sigma(F)\subset B^{2l+p}:=D^p_-\times D^{2l-1}_-\times[\frac13,\frac23]$. In the smooth category since $2(3l+p+1)\ge 3(p+1)+2(2l-1)+4$, we may assume that F is a smooth embedding outside B^{2l+p}. Let

$$M=\mathbb{R}^{3l+p}\times I-\operatorname{Int}R_{\mathbb{R}^{3l+p}\times I}(F(T^{p,2l}\times I-\operatorname{Int}B^{2l+p}),F\partial B^{2l+p}).$$

Observe that $F|_{B^{2l+p}}:B^{2l+p}\to M$ is a proper map whose restriction to the boundary is an embedding. Since $T^{p,2l}\times I$ is homologically $(p-1)$-connected, by Alexander duality M is homologically $(l+p-1)$-connected. Since M is simply-connected, it follows that M is $(l+p-1)$-connected.

Since $\beta(F)=0$, by Alexander duality it follows that $[FC]=0\in H_{l+p}(M;\mathbb{Z}_{(l)})$. Since $[FC]=0$, M is $(l+p-1)$-connected and $3l+p+1\ge 2l+p+3$, by the completeness theorem [Hu70', Theorem 2, Ha84, Theorem 4] it follows that $F|_{B^{2l+p}}$ is homotopic rel ∂B^{2l+p} to an embedding F'. Extending F' over N by F we obtain an embedding F'. Thus we obtain the required almost concordance from F to a concordance. $\square$

Reduction Lemma 6.1. *For $q=2l-1$, $m=3l+p$ and $p<l\ge 2$ the following conditions are equivalent:*

(G) $\beta(G)=0$ for every almost embedding $G:T^{p,q+1}\to S^{m+1}$;

(F) for almost concordances F between embeddings $f_0,f_1:T^{p,q}\to\mathbb{R}^m$ almost concordant to the standard embedding, the invariant $\beta(F)$ depends only on f_0 and f_1.

Proof that (F) implies (G). By the Standardization Lemma 2.1 and the assertion in the proof of the Triviality Criterion in §3 (or just analogously to the Standardization Lemma 2.1) we may assume that G is standardized, $G(S^p\times\frac12 D^{q+1}_+)\subset\frac12 D^{m+1}_+$ is the standard embedding and $G(S^p\times(D^{q+1}_+-\frac12 D^q_+))\subset D^{m+1}_+-\frac12 D^{m+1}_+$. Thus the restriction $F:S^p\times(D^{q+1}_+-\frac12 D^{q+1}_+)\to D^{m+1}_+-\frac12 D^{m+1}_+$ of G is an almost concordance between standard embeddings. So $\beta(G)=\beta(F)=0$. $\square$

Proof that (G) implies (F). For $s=0,1$ let F_s be any almost concordances from f_s to the standard embedding. Given an almost concordance F between f_0 and f_1 there is an almost concordance $F'=\overline{F}_0\cup F\cup F_1$ between standard embeddings. This F' can be 'capped' to obtain an almost embedding $G:T^{p,q+1}\to S^{m+1}$ without new self-intersections. Hence by the additivity of the β-invariant Theorem 2.9 we have $0=\beta(G)=\beta(F')=\beta(F)+\beta(F_0)-\beta(F_1)$. Thus $\beta(F)=\beta(F_1)-\beta(F_0)$ is independent on F. $\square$

The Additivity Theorem 2.10.b is proved analogously to the following absolute version.

Proof that $\beta(F+F')=\beta(F)+\beta(F')$ for almost embeddings $F,F':T^{p,2l}\to\mathbb{R}^{3l+p+1}$ and $0<p<l$. (Recall that for $p=l-1$ in the DIFF or AD category the sum is not necessarily

well-defined and we denote by $F+F'$ any sum.) Since $p<l$, we have $3l+p+1\ge 2p+2l+2$. Hence we may assume that F and F' are standardized. Then we may assume that the supports of C_F and $C_{F'}$ are in the ball $S^p\times D^{2l}_+\cap T^{p,2l}_-$.

Denote $m:=3l+p+1$. Since R_m and R_{2l} are isotopic to the identity maps of $\mathbb{R}^m$ and of S^{2l}, they do not change orientations. Hence $[\Sigma(F+F')]=[\Sigma(F)]+(\mathrm{id}\, S^p\times R_{2l})[\Sigma(F')]$. So we can take

$$C_{F+F'}:=C_F+(\mathrm{id}\, S^p\times R_{2l})C_{F'}\quad\text{and}\quad D_{F+F'}:=D_F+R_mD_{F'}.$$

We may assume that the supports of D_F and $D_{F'}$ are in $\mathbb{R}^m_+$. Then

$$\beta(F+F')=D_{F+F'}\cap[(F+F')(x\times S^{2l})]=(D_F+R_mD_{F'})\cap[F(x\times D^{2l}_+)+R_mF'(x\times D^{2l}_+)]=$$

$$=D_F\cap[F(x\times D^{2l}_+)]+R_mD_{F'}\cap[R_mF'(x\times D^{2l}_+)]=\beta(F)+\beta(F').\qquad\square$$

Note that for $p\le l-2$ the equality $\beta(F+F')=\beta(F)+\beta(F')$ holds for *general* almost concordances F and F' (for which f_0 and f_1 are not supposed to be almost concordant to the standard embedding).

An alternative proof that for $p\le l-2$ (G) implies that for almost concordances F between arbitrary embeddings $f_0,f_1:T^{p,q}\to\mathbb{R}^m$ the invariant $\beta(F)$ depends only on f_0 and f_1. Let F and F' be almost concordances between embeddings $f_0,f_1:T^{p,q}\to\mathbb{R}^m$. Then $F-F'$ is an almost concordance between embeddings isotopic to the standard embedding. Hence $F-F'$ can be 'capped' to obtain an almost embedding $G:T^{p,q+1}\to S^{m+1}$ without new self-intersections. So by the Additivity Theorem 2.10.b we have $\beta(F)-\beta(F')=\beta(F-F')=\beta(G)=0$. $\square$

The independence on C for $p=l-1$ and arbitrary f_0,f_1 cannot be reduced to the case when f_0 and f_1 are almost concordant to the standard embedding (as above) because $\beta(F+F')=\beta(F)+\beta(F')$ is not proved for $p=l-1$ and *general* almost concordances F and F' .

Triviality Lemma 6.2. *Suppose that either $1\le p\le l-2$ and $l+1$ is not a power of 2, or $1=p=l-1$. Then $\beta(G)=0$ for every smooth almost embedding $G:T^{p,2l}\to\mathbb{R}^{3l+p+1}$.*

Non-triviality Lemma 6.3. *For $1\le p<l\in\{3,7\}$ there exists a smooth almost embedding $G:T^{p,2l}\to\mathbb{R}^{3l+p+1}$ such that $\beta(G)=1$.*

We postpone the proof of these lemmas. Note that the Triviality Lemma 6.2 is true for $p=0\le l-2$.

The independence in the β-invariant Theorem 2.9 follows from $(G)\Rightarrow(F)$ of the Reduction Lemma 6.1 and the Triviality Lemma 6.2.

The triviality in the β-invariant Theorem 2.9 follows from $(F)\Rightarrow(G)$ of the Reduction Lemma 6.1, the Non-Triviality Lemma 6.3 and the completeness of β-invariant.

For the proof of the Triviality Lemma 6.2 and the Non-triviality Lemma 6.3 we need the Restriction Lemma 5.2 of §5 and so the condition $3l+p+1\ge 2p+2l+2$, i.e. $p<l$. More specifically, for the Non-triviality Lemma 6.3 we only need the definition of $\overline{\mu}$, for the Triviality Lemma 6.2 in the case when l is even we only need the definitions of $\overline{\mu}$ and $\overline{\nu}$ together with the exactness at $\overline{KT}^m_{p,q}$, and for the Triviality Lemma 6.2 in the general case we need the upper two lines of the Restriction Lemma 5.2 except the exactness at $\pi_{p+q}(S^{m-q-1})$.

From now until the end of this section we work in the smooth category which we omit from the notation.

Proof the Non-triviality Lemma 6.3. Consider the following diagram, in which H is the Hopf invariant.

$$\begin{array}{ccc} \pi_{2l+p}(S^{l+p}) & \underset{\overline{\mu}}{\longrightarrow} & \overline{KT}^{3l+p+1}_{p,2l} \\ \Big\downarrow \Sigma^\infty & & \Big\downarrow \beta \\ \pi^S_l & \underset{H}{\longrightarrow} & \mathbb{Z}_2 \end{array}$$

The diagram is (anti)commutative analogously to [Ko88, Theorem 4.8]. Since $p \geq 1$, it follows that the group $\pi_{2l+p}(S^{l+p})$ is either stable or metastable, so Σ^∞ is epimorphic. Since $l \in \{3, 7\}$, it follows that H is epimorphic. Hence β is epimorphic. $\square$

Proof of the Triviality Lemma 6.2. Since $2(3l+p+1) \geq 3\cdot 2l+4$, we can identify $\pi_{2l}(V_{l+p+1,p})$ with $KT^{3l+p+1}_{p,2l,+}$ by the isomorphism τ_+ of Lemma 5.1 which we omit from notation.

The set $\overline{KT}^{3l+p+1}_{p,2l}$ is a group for $p \leq l-2$ or $1 = p = l-1$ by Group Structure Lemma 2.2 and the Almost Smoothing Theorem 2.3.

In this paragraph assume that l *is even.* By the Additivity Theorem 2.10.b $\beta : \overline{KT}^{3l+p+1}_{p,2l} \to \mathbb{Z}$ is a homomorphism. So it suffices to prove that its domain is finite. Consider the following exact sequence given by the second line of the diagram from the Restriction Lemma 5.2 for $m = 3l+p+1$ and $q = 2l$:

$$\pi_{p+2l}(S^{l+p}) \underset{\overline{\mu}}{\longrightarrow} \overline{KT}^{3l+p+1}_{p,2l} \underset{\overline{\nu}}{\longrightarrow} \pi_{2l}(V_{l+p+1,p}).$$

The domain of $\overline{\mu}$ is finite for l even. The range of $\overline{\lambda}$ is finite for $l \geq 2$. (The finiteness is proved by the induction on p: for $p = 1$ and $l \geq 2$ the group $\pi_{2l}(S^{l+1}) \cong \pi^S_{l-1}$ is finite; the inductive step follows by applying exact sequence $\pi_{2l}(V_{l+p,p-1}) \to \pi_{2l}(V_{l+p+1,p}) \to \pi_{2l}(S^{l+p})$.) Hence $\overline{KT}^{3l+p+1}_{p,2l}$ is finite.

In this paragraph assume that l *is odd and* $l+1$ *is not a power of 2.* Consider the first two lines of the diagram from the Restriction Lemma 5.2 for $m = 3l+p+1$ and $q = 2l$ (except of the very left column). By Σ^p we denote *the right* Σ^p of the diagram, which maps the metastable group to the stable group. By μ'' we denote the map from this metastable group to the right (which is not shown on the diagram). Then $\langle[\iota_l, \iota_l]\rangle = \{[\iota_l, \iota_l], 0\}$ and $w_{l,p} = \mu''[\iota_l, \iota_l] \neq 0$ [Os86]. Hence by exactness $[\iota_l, \iota_l] \notin \operatorname{im}\lambda''$, so $\operatorname{im}\lambda'' \cap \ker\Sigma^p = 0$. Hence for any $x \in \overline{KT}^{3l+p+1}_{p,2l}$ we have

$$\overline{\lambda}\overline{\nu}(x) = 0 \quad\Rightarrow\quad \lambda''\overline{\nu}(x) \in \ker\Sigma^p \quad\Rightarrow\quad \lambda''\overline{\nu}(x) = 0 \quad\Rightarrow\quad \overline{\nu}(x) = \nu''x' \text{ for some } x' \quad\Rightarrow$$

$$\Rightarrow\quad \overline{\nu}(x - \tau(x')) = 0 \quad\Rightarrow\quad x - \tau(x') = \overline{\mu}(y) \text{ for some } y.$$

So $\overline{KT}^{3l+p+1}_{p,2l} = \operatorname{im}\tau + \operatorname{im}\overline{\mu}$. Since $\beta\tau = 0$ and $\beta\overline{\mu} = H\Sigma^\infty = 0$, the Lemma follows. $\square$

7. Proof of the Almost Smoothing Theorem 2.3

We need the following definitions and results. Recall from the beginning of §2 the convention concerning PL and PD categories. Recall that $T^{p,q}_+ = D^p_+ \times S^q$. Let $i : T^{m-q,q}_+ \to \mathbb{R}^m$ and $D^p \subset D^{m-q}$ be the standard inclusions. An embedding $g : T^{p,q}_+ \to \mathbb{R}^m$ is called *fiberwise* if

$$g(0, x) = i(0, x) \quad\text{and}\quad g(D^p \times x) \subset i(D^{m-q} \times x) \quad\text{for each}\quad x \in S^q.$$

Analogously a fiberwise concordance is defined.

Fibering Lemma 7.1. *For $m \ge p+q+3$ every PL embedding $f : T^{p,q}_+ \to \mathbb{R}^m$ is PL isotopic to a fiberwise PL embedding and every PL concordance between fiberwise PL embeddings $T^{p,q}_+ \to \mathbb{R}^m$ is PL isotopic relative to the ends to a fiberwise PL concordance.*

Note that codimension 0 analogue of the Fibering Lemma 7.1 is false [Hi66, Corollary B].

Slicing Lemma 7.2. *Let X be a finite simplicial n-complex, $m-n \ge 3$ and $s : X \to \mathbb{R}^m$ a simplicial map for some triangulation T of $\mathbb{R}^m$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that if a PL embedding $f : X \to \mathbb{R}^m$ is δ-close to s, then f is PL ε-ambient isotopic to a PL embedding $h : X \to \mathbb{R}^m$ such that $h^{-1}(\sigma) = s^{-1}(\sigma)$ for each cell σ of the dual to T cell-subdivision of $\mathbb{R}^m$* [Me02, Lemma 4.1].

Proof of the Fibering Lemma 7.1. (This proof appeared in a discussion of the author with S. Melikhov.) We prove only the assertion on embedding, the assertion on concordance is proved analogously. By the Zeeman Unknotting Spheres Theorem [Ze62] we may assume that $f(0,x) = i(0,x)$ for each $x \in S^q$. By making an isotopy of f which 'shrinks' $f(D^p \times x)$ we may assume that $f(T^{p,q}_+) \subset i(T^{m-q,q}_+)$ and $f(a,x) = i(b,y)$ implies that $|x,y| < \delta$.

Apply the Slicing Lemma 7.2 to the standard embedding s and the embedding f to obtain an embedding h. Then for some cell-subdivision T_1 of S^q we have $h(D^p \times \sigma_1) \subset i(D^{m-q} \times \sigma_1)$ for each dual cell σ_1 of T_1 (because $D^{m-q} \times \sigma_1$ is a dual cell for certain triangulation of $\mathbb{R}^m$).

By induction on k we may assume that $h(D^p \times x) \subset D^{m-q} \times x$ for points x of the k-skeleton of T_1. Recall the relative k-Concordance Implies k-Isotopy Theorem:

if an embedding $g : D^p \times D^k \to D^{m-q} \times D^k$ on $D^p \times \partial D^k$ commutes with the projection onto the second factor, and $g(0,x) = i(0,x)$ for each $x \in D^k$, then the embedding is isotopic relative to $D^p \times \partial D^k \cup 0 \times D^k$ to an embedding commuting with the projection onto the second factor.

Applying this result for each $(k+1)$-cell of T_1 independently we obtain an isotopy of h relative to the k-skeleton to an embedding f' such that $f'(D^p \times x) \subset D^{m-q} \times x$ for points x of the $(k+1)$-skeleton of T_1. This proves the inductive step. For $k = n$ we obtain a fiberwise PL embedding. $\square$

Proof of the Almost Smoothing Theorem 2.3. Since any embedding $S^0 \to S^{m-q-1}$ is canonically isotopic to an embedding whose image consists of two antipodal points, by the Fibering Lemma 7.1 it follows that

every PL embedding $T^{1,q}_+ \to \mathbb{R}^m$ is PL isotopic to a smooth embedding, and every PL concordance between smooth embeddings $T^{1,q}_+ \to \mathbb{R}^m$ is PL isotopic to a smooth concordance.

Now the Almost Smoothing Theorem 2.3 follows because

for $m \ge q + \frac{3p+5}{2}$ the group $KT^m_{p,q,AD}$ is isomorphic to that of PL embeddings $T^{p,q} \to \mathbb{R}^m$ which are smooth embeddings on $T^{p,q}_+$, up to PL isotopy which is a smooth isotopy on $T^{p,q}_+ \times I$.

Let us prove the latter statement. By [Ha67, Ha] for such an embedding $T^{p,q} \to \mathbb{R}^m$ the obstructions to extension of smoothing from $T^{p,q}_+$ to $T^{p,q}_+ \bigcup\limits_{\partial D^p_- \times D^q_+} D^p_- \times D^q_+$ are in

$$H^{i+1}(D^p_- \times D^q_+, \partial D^p_- \times D^q_+; C^{m-q}_i) \cong H^{i+1}(S^p; C^{m-q}_i) \cong 0 \quad \text{for} \quad 2(m-q) \ge 3p+4.$$

Therefore every embedding as above is *almost smoothable*, i.e. is PL concordant to an almost smooth embedding.

Analogously, for $2(m+1-q) \ge 3(p+1)+4$ every PL isotopy between smooth embeddings, which is a smooth isotopy on $T^{p,q}_+ \times I$, is *almost smoothable*, i.e is PL concordant relative to the ends to an almost smooth concordance. $\square$

Appendix: Manifolds with boundary.

We show that the dimension restriction is sharp in the completeness results for α-invariant of manifolds with boundary [Ha63, 6.4, Sk02, Theorems 1.1.$\alpha\partial$ and 1.3.$\alpha\partial$]. This is (much simpler) non-closed analogue of the Whitehead Torus Example 1.7.

Filled-tori Theorem 7.3. *(a) $KT^m_{p,q,+,PL} \cong \pi_q(V^{PL}_{m-q,p})$ for $m \geq p+q+3$, where the PL Stiefel manifold $V^{PL}_{m-q,p}$ is the space of PL embeddings $S^{p-1} \to S^{m-q-1}$.*

(b) The map $\alpha^m(T^{1,q}_+)$ is not surjective if $m \geq q+4$ and $\Sigma^\infty : \pi_q(S^{m-q-1}) \to \pi^S_{2q+1-m}$ is not epimorphic (concrete examples can be found from the table in [Sk02, Example 1.4.s]).

(c) The map $\alpha^m(T^{1,q}_+)$ is not injective if $m \geq q+4$ and $\Sigma^\infty : \pi_q(S^{m-q-1}) \to \pi^S_{2q+1-m}$ is not monomorphic, e.g. for $q = 2l-1$, $m = 3l$, $l \geq 4$ and $l \neq 7$.

(d) The map $\alpha^{3l+p-1}_{DIFF}(T^{p,2l-1}_+)$ is not injective for each even l and $1 \leq p < l$.

Proof. Part (a) follows by the Fibering Lemma 7.1.

Since $V^{CAT}_{m-q,1} \simeq S^{m-q-1}$, by part (a) and the assertions on τ and $\alpha\tau$ of the Torus Theorem 2.8 it follows that

$$KT^m_{1,q,+} \cong \pi_q(S^{m-q-1}) \quad \text{and} \quad \pi^{m-1}_{eq}(\widetilde{T^{1,q}_+}) \cong \pi^S_{2q+1-m}.$$

Analogously to the proof of the Torus Theorem 2.8, α corresponds to the suspension under the above isomorphisms. This implies parts (b) and (c).

Part (d) follows because

$$KT^{3l+p-1}_{p,2l-1,+,DIFF} \cong \pi_{2l-1}(V_{l+p,p}) \quad \text{is infinite and} \quad \pi^{3l+p-2}_{eq}(\widetilde{T^{p,2l-1}_+}) \cong \Pi^{3l+p-2}_{p-1,2l-1} \quad \text{is finite}$$

by the asserion on τ of the Torus Theorem 2.8 and [Sk02, Lemma 7.3.a]. The infiniteness of $\pi_{2l-1}(V_{l+p,p})$ for l even and $1 \leq p < l$ is proved by the induction on p [cf. Sk02, Proof of Lemma 7.3.a]. The base $p = 1$ is due to Serre. The inductive step is proved using the following exact sequence for $p > 1$:

$$\dots \to \pi_{2l}(S^{l+p-1}) \to \pi_{2l-1}(V_{l+p-1,p-1}) \to \pi_{2l-1}(V_{l+p,p}) \to \pi_{2l-1}(S^{l+p-1}) \to \dots \quad \square$$

8. Proofs of the Main Theorems 1.3 and 1.4 in the smooth category

Smoothing Theorem 8.1. *For $m \geq 2p+q+3$ there is an exact sequence of groups:*

$$\dots \to KT^{m+1}_{p,q+1,AD} \stackrel{\sigma_{q+1}}{\to} C^{m-p-q}_{p+q} \stackrel{\zeta}{\to} KT^m_{p,q,DIFF} \stackrel{\text{forg}}{\to} KT^m_{p,q,AD} \stackrel{\sigma_q}{\to} C^{m-p-q}_{p+q-1} \to \dots$$

For $m \geq p+q+3$ there is exact sequence of *sets* as in the Smoothing Theorem 8.1.

Definition of σ_q. Take the codimension zero ball $B^{p+q} \subset T^{p,q}$ from the definition of an almost smooth embedding, and an almost smooth embedding $f : T^{p,q} \to \mathbb{R}^m$. By making a PL isotopy fixed outside $\operatorname{Int} B^{p+q}$ we may assume that f is smooth outside a fixed single point [Ha67], cf. beginning of [BH70, Bo71]. Consider a small smooth oriented $(m-1)$-sphere Σ with the center at the image of this point. Take the natural orientation on the $(p+q-1)$-sphere $f^{-1}\Sigma$. Let $\sigma(f)$ be the isotopy class of the abbreviation $f^{-1}\Sigma \to \Sigma$ of f. By [BH70, Bo71] $\sigma(f)$ is well-defined, i.e. independent on choices in the definition.

For all applications of the Smoothing Theorem 8.1 but Theorem 1.2DIFF the exactness at $KT^m_{p,q,DIFF}$ is sufficient.

Proof of the exactness at $KT^m_{p,q,DIFF}$. Clearly, ζ is a homomorphism. By the PL Unknotting Spheres Theorem $\mathrm{forg}\circ\zeta = 0$.

Take a smooth embedding $f : T^{p,q} \to S^m$ such that $\mathrm{forg}\, f = 0$. Let F be an almost smooth isotopy from f to the standard embedding. Analogously to the above definition of σ_q it is defined a complete obstruction $\sigma(F) \in C^{m-p-q}_{p+q}$ to *smoothability* of F, i.e. to the existence of an almost smooth concordance from F to a smooth embedding [Ha67, Ha, BH70, Bo71].

Take a smooth embedding $g : S^{p+q} \to S^m$ representing $-\sigma(F)$. From $\Sigma g : S^{p+q+1} \to S^{m+1}$ we can easily construct an almost smooth isotopy $G : S^{p+q} \times I \to S^m \times I$ between g and the standard embedding such that $\sigma(G) = [g] = -\sigma(F)$. Then $F\#G$ is an almost smooth concordance from $f\#g = f - \zeta(\sigma(F))$ to the standard embedding. We have $\sigma(F\#G) = \sigma(F) + \sigma(G) = 0$, so $F\#G$ is smoothable. Therefore $f = \zeta(\sigma(F)) \in \mathrm{im}\,\zeta$. $\square$

The exactness at $KT^m_{p,q,AD}$ is clear.

Proof of the exactness at C^{m-p-q}_{p+q}. The proof appeared in a discussion with M. Skopenkov. Let X be the set of proper almost smooth embeddings $S^p \times D^{q+1} \to D^m$ up to proper almost smooth isotopy (not necessarily fixed on the boundary). Consider the sequence

$$KT^{m+1}_{p,q+1,AD} \xrightarrow{u} X \xrightarrow{r} KT^m_{p,q,DIFF}.$$

Here r is the restriction to the boundary. In order to define u, take an almost smooth embedding $f : T^{p,q+1} \to \mathbb{R}^{m+1}$. By the Standardization Lemma 2.1 on embeddings we may assume that f is standardized. Let $u(f) = f|_{T^{p,q+1}_+} : T^{p,q+1}_+ \to \mathbb{R}^{m+1}_+$. By the Standardization Lemma 2.1 on concordances this is well-defined.

By the Triviality Criterion of §3 $ru = 0$. Take an almost smooth embedding $f_+ : T^{p,q+1}_+ \to \mathbb{R}^{m+1}_+$ such that $rf = 0$. Capping a smooth isotopy from rf to the standard embedding we obtain a smooth embedding $f_- : T^{p,q+1}_- \to \mathbb{R}^{m+1}_-$ agreeing with f on the boundary. The embedding f_- is smoothly isotopic to the standard embedding by the Uniqueness Lemma 8.2 below. Since this isotopy is ambient, $f_+ \cup f_-$ is isotopic to a standardized embedding $f : T^{p,q+1} \to \mathbb{R}^{m+1}$ whose restriction to $T^{p,q+1}_+$ is properly smoothly isotopic to f_+. Hence $f_+ = uf$. Thus the (r,u)-sequence is exact.

There is a 'boundary connected sum with cone' action $\zeta' : C^{m-p-q}_{p+q} \to X$. Analogously to the above definition of σ_q it is defined a complete obstruction $\sigma : X \to C^{m-p-q}_{p+q}$ to smoothing. Clearly, $\zeta'\sigma = \mathrm{id}_X$ and $\sigma\zeta' = \mathrm{id}_{C^{m-p-q}_{p+q}}$. Thus ζ' and σ are 1–1 correspondences. Clearly, $r = \zeta'\zeta$ and $\sigma_{q+1} = \zeta' u$. Hence the sequence of the Smoothing Theorem 8.1 is exact at C^{m-p-q}_{p+q}.

Uniqueness Lemma 8.2. *If $m \ge 2p + q + 3$, then each two smooth proper embeddings $S^p \times D^q \to D^m$ are smoothly proper isotopic.*

Proof. Use the assertion in the proof of the Triviality Criterion in §3. Since $m \ge p+q+3$, the restriction $f|_{S^p\times(D^{q+1}_+ - \frac12 D^{q+1}_+)}$ is smoothly isotopic relative to the boundary to a smooth isotopy $F : S^p \times \partial D^{q+1}_+ \times I \to \mathbb{R}^m \times I$ between $F_0 = f|_{S^p\times\partial D^{q+1}_+}$ and the standard embedding F_1. For

$$\tau \in [0,1] \quad \text{define} \quad \tau_* : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1} \quad \text{by} \quad \tau(x_0, x_1, \dots, x_m) := (x_0 - \tau, x_1, \dots, x_m).$$

Take an isotopy Φ from $F \cup f|_{S^p\times\frac12 D^{q+1}_+}$ to the standard embedding such that the image of Φ_τ is the union of $\tau_* f(S^p \times \frac12 D^{q+1}_+)$ and $\tau_* F_t$ for $t \ge \tau$. $\square$

Proof of the Main Theorem 1.3.b in the smooth case. Follows by the AD case (proved in §2 together with the PL case), the exactness at $KT^m_{p,q,DIFF}$ of the Smoothing Theorem 8.1 and the equality $C^{2k+2}_{4k+2}=0$ [Ha66, 8.15, Mi72, Theorem F] because *the map* forg *is surjective for* $p=1$*,* $q=2k+1$ *and* $m=6k+4$. The surjectivity of forg follows because the composition of forg with $\tau\oplus\omega$ is surjective by the Realization Theorem 2.4.b.

We denote by $\widehat{h}$ certain quotient homomorphism of a homomorphism h.

Proof of the Main Theorem 1.4.DIFF. The existence of the exact sequence follows from the Smoothing Theorem 8.1. The surjectivity of the map forg follows because the composition of forg with $\tau\oplus\omega$ is surjective by the Realization Theorem 2.3.b.

Let us prove that *for* l *odd the exact sequence splits, i.e. the above map* forg *has a right inverse.* By the Main Theorem 1.4.AD and the Realization Theorem 2.4.b we can identify

$$KT^{3l+p}_{p,2l-1,AD}\quad\text{with}\quad\frac{\pi_{2l-1}(V_{l+p+1,p+1})}{w_{l,p}}\oplus\widehat{\mathbb{Z}_2}\quad\text{via the isomorphism}\quad\widehat{\tau}_{AD}\oplus\widehat{\omega}_{AD}.$$

By the Relation Theorem 2.7 $\tau_{DIFF}(w_{l,p})=0$. Then $\tau_{DIFF}\oplus\omega_{DIFF}$ factors through a right inverse of forg. $\square$

Proof of the Main Theorem 1.3.aDIFF. Follows from the Main Theorem 1.4.AD,DIFF and the Smoothing Theorem 8.1 because *for* $p=1$*,* $q=4k\ge 8$ *and* $m=6k+1$ *the above map* forg *has a right inverse.*

Let us prove the latter statement. We omit the DIFF category from the notation. Let $l=2k$. By the Realization Theorem 2.4.a in the AD category we can identify

$$KT^{3l+1}_{1,2l-1,AD}\quad\text{with}\quad\pi^S_{l-2}\oplus\frac{\pi_{2l-1}(S^l)\oplus\mathbb{Z}}{[\iota_l,\iota_l]\oplus 2}\quad\text{via the isomorphism}\quad\tau^1_{AD}\oplus(\widehat{\tau^2_{AD}\oplus\omega_{AD}}).$$

By the τ-relation of the Relation Theorem 2.7 we have $\tau^2_{AD}[\iota_l,\iota_l]=2\omega_{AD}$. Then by smoothing theory [BH70, Bo71] there is an element $\psi\in C^{l+1}_{2l}$ such that $\tau^2[\iota_l,\iota_l]-2\omega=\zeta(\psi)$. Since $[\iota_l,\iota_l]$ is not divisible by 2 for l even, $l\ge 4$ [Co95], it follows that $[\iota_l,\iota_l]\oplus 2$ is primitive. Hence there is a homomorphism

$$\varphi:\pi_{2l-1}(S^l)\oplus\mathbb{Z}\to C^{l+1}_{2l}\quad\text{such that}\quad\varphi([\iota_l,\iota_l]\oplus 2)=\psi.$$

Then $\tau^1\oplus[(\tau^2\oplus\omega)-\zeta\varphi]$ factors through a right inverse of forg. $\square$

Proof of Theorem 1.2DIFF. For $2m\ge 3q+2p+4$ we have $C^{m-p-q}_q=C^{m-p-q}_p=0$ [Ha66], so $KT^m_{p,q,PL}=KT^m_{p,q,AD}$ by smoothing theory [BH70, Bo71], cf. [Ha67, Ha].

For $2m\ge 3q+2p+4$ by the assertion on $\alpha\tau$ of the Torus Theorem 2.8 the map $\tau_{DIFF}(\alpha\tau_{DIFF})^{-1}\alpha_{PL}$ is a right inverse of forg (hence $\sigma_q=0$). If $2m=3q+2p+3$, then $m\ge 2p+q+3$ implies that $q\ge 2p+3$, so $\sigma_q=0$ by the Main Theorem 1.4DIFF. Thus Theorem 1.2DIFF follows from Theorem 1.2PL and the Smoothing Theorem 8.1.

Appendix: Remarks, conjectures and open problems

See some open problems in [Sk07, §3].

(0) Explicit constructions of §2 allow to prove the following remark (which does not follow immediately from the definition of inverse elements).

Symmetry Remark. *On* $KT^7_{1,3,PL}\cong KT^7_{1,3,AD}$ *we have* $\mathrm{id}=\sigma_1=-\sigma_3=\sigma_7$.

Proof. Let $a_k : S^k \to S^k$ be the antipodal involution. We have

$$\sigma_1\tau^1 = \tau^1(a_3 \circ \iota_3) = \tau^1 \quad \text{and} \quad \sigma_3\tau^1 = \sigma^3\tau^1(\Sigma\iota_2) = \tau^1(\sigma_3 \circ \iota_3) = \tau^1(-\iota_3) = -\tau^1.$$

Hence $\sigma_7\tau^1 = -\sigma_3\tau^1 = \tau^1$.

Let $\eta \in \pi_3(S^2) \cong \mathbb{Z}$ be the generator (i.e. the class of the Hopf map) and $h_0 : \pi_3(S^2) \to \mathbb{Z}$ the Hopf isomorphism. We have by [Po85, Complement to Lecture 6, (10)]

$$\sigma_1\tau^2 = \sigma_1^2\tau^2 = \tau^2(a_2 \circ \eta) = \tau^2(-\eta + [\iota_2, \iota_2]h_0(\eta)) = \tau^2(-\eta + 2\eta) = \tau^2.$$

Hence

$$\sigma_7\tau^2 = \sigma_1^1\tau^2 = \sigma_1\tau^2 = \tau^2 \quad \text{and} \quad \sigma_3\tau^2 = -\sigma_7\tau^2 = -\tau^2.$$

The relation for ω follows from the Symmetry Lemma of §4. $\square$

Note that $\omega_{0,AD}$ cannot be extended to a *smooth* embedding $T_+^{1,2l-1} \to \mathbb{R}_+^{3l+1}$ without connected summation with φ. Indeed, if there existed such an extension, then we could modify it by boundary surgery to a smoothly embedded $2l$-manifold bounding the connected sum of Borromean rings. Then surger certain circle in this manifold so as to obtain a smoothly embedded $2l$-disk bounding the connected sum of Borromean rings. The latter is impossible by [Ha62'].

(1) Describe possible homotopy types of complements to knotted tori. E. g. for an embedding $f : T^{1,3} \to S^7$ the complement is determined by an element of $\pi_4(S^2 \vee S^3) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$. Describe possible normal bundles of knotted tori, cf. [MR71]. E. g. prove that

$$\nu(\tau(\varphi)) \cong 1 \oplus \kappa^*\pi^*\varphi^*t, \quad \text{where} \quad S^q \xrightarrow{\varphi} V_{m-q,p+1} \xrightarrow{\pi} G_{m-q,p+1} \overset{\kappa}{\cong} G_{m-q,m-q-p-1} \to G_{m,m-p-1}$$

and t is the tautological bundle. Describe possible immersion classes of knotted tori.

(2) Which values can assume the linking coefficient of an embedding $T^{0,q} \to \mathbb{R}^m$, extendable to an embedding $T^{p,q} \to \mathbb{R}^m$? More generally, using the classification of links [Ha66', Hab86] and knotted tori, describe natural maps

$$KT^m_{p,q} \to KT^m_{p,q,+} \to KT^m_{p-1,q} \to KT^m_{0,q} \quad \text{and} \quad KT^m_{p,q} \to KT^{m+1}_{p,q}.$$

See [CR05]. Observe that the restriction $KT^m_{p,q,CAT} \to KT^m_{p,q,CAT,+} \cong \pi_q(V^{CAT}_{m-q,p})$ is a complete invariant for regular homotopy. For $p = 1$ this invariant is the linking coefficient ν defined in §5 [Sk02, Decomposition Lemma 7.1].

From the Filled-tori Theorem 7.3.a and Theorem 1.2 it follows that

The restriction map $r^m_{p,q} : KT^m_{p,q,+} \to KT^m_{p-1,q}$ *is a 1–1 correspondence if*
$2m \ge 3q + 2p + 4$ *in the PL category, or*
$p - 1 \le q$ *and* $m \ge \max\{(3p + 3q + 1)/2, 2p + q + 2\}$ *in the smooth category.*

Let us sketch an idea of a possible alternative direct proof of the first of these facts. Let us prove the surjectivity of r (the injectivity can be proved analogously). Take an embedding $f : T^{p-1,q} \to \mathbb{R}^m$. To each section of the normal bundle of f there corresponds an embedding $f' : T^{p-1,q} \times I \to \mathbb{R}^m$. The latter can be extended to an embedding $f'' : (T_+^{p,q} - B^{p+q}) \to \mathbb{R}^m$ by general position. We can always find a section such that the latter embedding extends to a map $f''' : T_+^{p,q} \to \mathbb{R}^m$ whose self-intersection set is contained in B^{p+q}. The latter map is homotopic to an embedding relative to $T_+^{p,q} - B^{p+q}$ analogously to [Sk02, Theorem 2.2.q].

A *fiberwise embedding* $T^{p,q} \to \mathbb{R}^m$ is defined analogously to that of $T_+^{p,q} \to \mathbb{R}^m$. It would be interesting to find a direct proof of the following corollary of Filled-tori Theorem 7.3.d and Theorem 1.2.

For $m \ge \max\{2p+q+2, 3q/2+p+2\}$ every PL embedding $f : T^{p,q} \to \mathbb{R}^m$ is PL isotopic to a fiberwise PL embedding and every PL isotopy between fiberwise PL embeddings is PL isotopic relative to the ends to a fiberwise PL isotopy.

(3) We conjecture that if

$$f, g : T^{p,q} \to S^m \quad \text{are embeddings,} \quad M = S^m - R(f(S^p \vee S^q)), \quad D^{p+q} = T^{p,q} - f^{-1}(\operatorname{Int} M),$$

$$f = g \text{ on } f^{-1}R(f(S^p \vee S^q)) \quad \text{and} \quad g(D^{p+q}) \subset M, \quad \text{then} \quad [f|_{D^{p+q}}] = [g|_{D^{p+q}}] \in \pi_{p+q}(M, \partial M).$$

(4) Construct examples of non-surjectivity of $\alpha^{3l+p+1}(T^{p,2l})$.

We conjecture that $\beta^{3l+p-1}(T^{p,2l-1}_+)$ is not injective and so the dimenson restiction in the injectivity of [Sk02, Theorem 1.1.$\beta\partial$] is sharp.

(5) We conjecture that the exact sequence of the Main Theorem 1.4 in the DIFF case splits also for l even. The proof of this conjecture requires the smooth case of the τ-relation for l even. A possible approach to prove this is to follow the proof of the completeness of β-invariant [Ha84] and prove that for our particular almost concordance $\sigma_1\Omega_1 \cup \overline{\Omega}_1$ we can obtain a *smooth* concordance from $\sigma_1\omega_1$ to ω_1. Or else the required assertion is implied by the fact that every almost smooth embedding $T^{1,2l} \to \mathbb{R}^{3l+2}$ is smoothable. We also need to smoothen the relation $\tau_p(w_{l,p}) = \omega_p + \sigma_p\omega_p$.

Let us prove that *the map* forg *from the Smoothing Theorem 8.1 has a right inverse if $2m = 3q + 2p + 3$, $q = 2l - 1$, $p < l$, l is even and $w_{l,p}$ is not divisible by 2.*

Indeed, by the AD case of the Main Theorem 1.4 we can identify $KT^{3l+p}_{p,2l-1,AD}$ with $\frac{\pi_{2l-1}(V_{l+p+1,p+1})\oplus\mathbb{Z}}{w_{l,p}\oplus 2}$ via the isomorphism $\widehat{\tau_{AD} \oplus \omega}_{AD}$. Recall that $\tau(w_{l,p}) = 2\omega_p$ in the almost smooth category. Then there is an element $x \in C^{l+1}_{2l+p-1}$ such that $\tau(w_{l,p}) - 2\omega_p = \zeta(x)$ in the smooth category. Since $w_{l,p}$ is not divisible by 2, it follows that $w_{l,p} \oplus 2$ is a primitive element. Hence there is a homomorphism $\varphi : \pi_{2l-1}(V_{l+p+1,p+1}) \oplus \mathbb{Z} \to C^{l+1}_{2l+p-1}$ such that $\varphi(w_{l,p} \oplus 2) = x$. Then $(\tau \oplus \omega) - \zeta\varphi$ factors through a right inverse of forg.

(6) We conjecture that the map $\bar{\alpha}'\tau$ from Retsriction Lemma 5.2 is not epimorphic and $\beta \equiv 0$ when $w_{l,p} = 0$ (in particular, for $l = 15$).

(7) We conjecture that $\pi^{m-1}_{eq}((S^p \times S^q - \operatorname{Int} D^{p+q})\tilde{\ }) \cong \Pi^{m-1}_{p-1,q}$ for $m \ge 2p + q + 2$.

(8) We conjecture that *the forgetful map $KT^{3l+p}_{p,2l-1,AD} \to KT^{3l+p}_{p,2l-1,PL}$ is an isomorphism for* $0 < p < l$, or, equivalently, that *the independence in the β-invariant Theorem 2.9 hold in the PL case*, or, equivalently, that *the Triviality Lemma 6.2 hold in the PL case.* If this conjecture is true, then there is less indeterminancy in the Main Theorem 1.4 for the PL case, and the Whitehead Torus Example 1.7 is true for embeddings $T^{p,2l-1} \to \mathbb{R}^{3l+p}$ and $l + 1$ not a power of 2.

If $m \ge \frac{3q}{2} + p + 2$, then by smoothing theory the forgetful map $KT^m_{p,q,AD} \to KT^m_{p,q,PL}$ is an isomorphism (moreover, every PL embedding $T^{p,q} \to \mathbb{R}^m$ is almost smoothable and every PL isotopy between such embeddings is almost smoothable *relative to the boundary*).

The forgetful homomorphism $KT^{3l+p}_{p,2l-1,+,DIFF} \to KT^{3l+p}_{p,2l-1,+,PL}$ is an isomorphism by the assertion on τ of the Torus Theorem 2.8 because $2(3l + p) = 3(2l - 1) + 2(p - 1) + 5$. But this does not imply the conjecture because passage from PL isotopy to a smooth one is made by a homotopy which could have self-intersections. This also does not imply that every PL (almost) concordance F between smooth embeddings $T^{p,2l-1}_+ \to \mathbb{R}^{3l+p}$ is smoothable relative to the boundary, thus changing such an almost concordance to a smooth almost concordance can possibly change β-invariant. The obstruction to such a smoothing can be non-zero because this is an obstruction to smoothing but not just to an existence of a smooth

concordance between the same ends (or, in other words, to extension of a smooth embedding to $T^{p,2l}_+$ from the boundary not just existence of a smooth embedding).

The smoothing obstruction is that to extend smoothing given on a neighborhood of $x \times \partial D^{2l}$, to $x \times D^{2l}$. This obstruction splits into two parts. The first one is that to extend smoothing from $x \times \partial D^{2l}$ to $x \times D^{2l}$ and is in $C^{l+p+1}_{2l-1} = 0$ for $p > 0$. The second one is to extend a smooth p-frame from $x \times \partial D^{2l}$ to $x \times D^{2l}$ and is in $\pi_{2l-1}(V^{PL}_{l+p+1,p}, V_{l+p+1,p})$. The stabilization of the second obstruction in $\pi_{2l-1}(V^{PL}_{l+M+1,M}, V_{l+M+1,M}) \cong C^{l+1}_{2l-1}$ is perhaps the Haefliger smoothing obstruction [Ha].

The set of invariants of $\ker(KT^{3l+p}_{p,2l-1,PL} \to KT^{3l+p}_{p,2l-1,+,PL})$ depend on $KT^{3l+p+1}_{p,2l,+,PL}$ and so it is hard to compare it with the same kernel in the almost smooth category.

It would be interesting to understand how non-smoothable embeddings $T^{p,2l,+} \to \mathbb{R}^{3l+p+1}$ are constructed from a knot $S^{2l-1} \to S^{3l}$ (presently they are coming by the Miller isomorphism from homotopy groups of spheres), and to prove that they extend to a PL embedding $T^{p,2l} \to \mathbb{R}^{3l+p+1}$. In other words, it would be interesting to construct a map ζ' such that the sequence $\mathbb{Z}_{(l)} \cong C^{l+1}_{2l-1} \xrightarrow{\zeta'} KT^{3l+p}_{p,2l,AD} \to KT^{3l+p}_{p,2l,PL}$ is exact, cf. [CRS], and then to use this map to prove the independence of β-invariant in the PL case for $p > 1$. Note that multiplication of such a knot with S^p for $p << l$ gives an obstruction in C^{l+1}_{2l-1+p} not in C^{l+1}_{2l-1}, while multiplication of such a knot with D^p for $p \geq l-2$ gives an embedding isotopic to the standard embedding. For $p \neq 1$ this map is defined using not S^p but p-manifolds (or even p-cycles).

The forgetful homomorphism $\overline{KT}^{3l+p}_{p,2l-1,AD} \to \overline{KT}^{3l+p}_{p,2l-1,PL}$ is an isomorphism by the assertion on $\overline{\alpha}$ of the Torus Theorem 2.8 because $2(3l+p) = 3(2l-1)+2p+3$. But this does not imply the conjecture because we cannot apply 5-lemma to the forgetful map of exact sequences $\overline{KT}^{3l+p+1}_{p,2l} \to \mathbb{Z}_{(l)} \to KT^{3l+p}_{p,2l-1} \to \overline{KT}^{3l+p}_{p,2l-1} \to 0$, since forg : $\overline{KT}^{3l+p+1}_{p,2l,AD} \to \overline{KT}^{3l+p+1}_{p,2l,PL}$ is not proved to be epimorphic. We know that $\overline{\alpha}^{3l+p+1}_{p,2l,AD}$ is an isomorphism, but this does not imply that $(\overline{\alpha}_{AD})^{-1}\overline{\alpha}_{PL}$ is an inverse of forg, so again we cannot conclude that $\beta_{PL} = 0$.

The generalization of the above conjecture for arbitrary m, p, q is true if and only if the inclusion-induced homomorphism $\pi_q(V_{m-q,p}) \to \pi_q(V^{PL}_{m-q,p})$ is an isomorphism (as in the two Haefliger's examples below).

Recall some known facts on the PL Stiefel manifolds. For $p > q$ and $l > 2$ by [Ha66, 4.8, Ha, 10.2, 11.2] there is an exact sequence

$$\dots \to \pi_{q+1}(V^{PL}_{p+l,p}) \to C^l_q \to \pi_q(V_{p+l,p}) \xrightarrow{\rho^{PL}_q} \pi_q(V^{PL}_{p+l,p}) \to \dots .$$

Note that such a sequence is not exact for $p = 1$. If $p \geq q$ and $l > 2$ this and [Ha66] imply that *the inclusion-induced map* $\pi_q(V_{p+l,p}) \to \pi_q(V^{PL}_{p+l,p})$ *is an isomorphism for* $q \leq 2l-4$ *and an epimorphism for* $q = 2l-3$.

Denote $V = V_{l+p+1,p}$ and $V^{PL} = V^{PL}_{l+p+1,p}$. Consider the following part of the above exact sequence for $p \geq 2l$:

$$\pi_{2l+1}(V) \to \pi_{2l+1}(V^{PL}) \to C^{l+1}_{2l} \to \pi_{2l}(V) \to \pi_{2l}(V^{PL}) \to C^{l+1}_{2l-1} \to \pi_{2l-1}(V) \to \pi_{2l-1}(V^{PL}) \to 0.$$

Then

for l odd ρ^{PL}_{2l} is monomorphic and ρ^{PL}_{2l+1} is epimorphic (because $C^{l+1}_{2l} = 0$),
ρ^{PL}_{2l} is not epimorphic for l even and $p = 2l$ [Ha67, §5], and
ρ^{PL}_{2l+1} is not monomorphic for l odd and $p = 2l+1$ [Ha67, §5].

(9) Is it correct that $\beta(h)=\beta(f)+\beta(g,f(x\times S^{2l}))$ if the almost embedding $h:T^{p,2l}\to\mathbb{R}^{3l+p}$ is obtained from an almost embedding $f:T^{p,2l}\to\mathbb{R}^{3l+p}$ by connected summation with a map $g:S^{p+2l}\to\mathbb{R}^{3l+p}-f(x\times S^{2l})$, when the images of f and g are not assumed to be contained in disjoint balls? U. Koschorke kindly informed me that the difference $\beta(f)+\beta(g,f(x\times S^{2l}))-\beta(h)$ is formed by intersection in which we can choose the first and the second sheet, so the classification space is D^0 not $\mathbb{R}P^\infty$, thus such a difference is a simple invariant (but still it is not clear whether it is trivial or not).

(10) Note that the Fibering Lemma 7.1 does not immediately follow from the 'n-concordance implies n-isotopy' theorem because it is not clear why there exist balls $D^{m-q}\times D^q_\pm\subset\mathbb{R}^m$ such that

$$f(D^p\times D^q_\pm)\subset D^{m-q}\times D^q_\pm \quad\text{and}\quad D^{m-q}\times D^q_+\cap D^{m-q}\times D^q_-=D^{m-q}\times\partial D^q_+=D^{m-q}\times\partial D^q_-.$$

Such balls are constructed using the Slicing Lemma 7.2.

(11) **Proof of the Slicing Lemma 7.2.**

We present the proof from [Me02] with S. Melikhov's kind permission, because the proof is less technical in our situation. Consider the following statement.

(k,l) *Let X be a finite simplicial n-complex, $m-n\geq 3$ and $s:X\to\mathbb{R}^m$ a simplicial map for some triangulation T of $\mathbb{R}^m$. Let $k=\dim s(X)$, let C be the union of some top-dimensional dual cells in $\mathbb{R}^m$ and let l be the number of top-dimensional dual cells in* $\mathrm{Cl}(\mathbb{R}^m-C)$*. Then for each $\varepsilon>0$ there exists $\delta>0$ such that each PL embedding $f:X\to\mathbb{R}^m$ which is δ-close to s and such that $f^{-1}(\sigma)=s^{-1}(\sigma)$ for each dual cell $\sigma\subset C$, is PL ε-ambient isotopic keeping C fixed to a PL embedding $h:X\to\mathbb{R}^m$ such that $h^{-1}(\sigma)=s^{-1}(\sigma)$ for each cell σ of the dual to T cell-subdivision of $\mathbb{R}^m$.*

Then (i,0) and (0,j) are trivial for any i, j. Assuming that (i,j) is proved for $i<k$ and arbitrary j, and also for $i=k$ and $j<l$, let us prove (k,l). We shall reduce it to the following Edwards' lemma. Consider the projections

$$\Pi_1:Q\times\mathbb{R}\to Q,\qquad \Pi_2:Q\times\mathbb{R}\to\mathbb{R}\qquad\text{and}\qquad \pi:X\times[-1,1]\to[-1,1]\subset\mathbb{R}.$$

Edwards' Slicing Lemma. *For each $\varepsilon>0$ and a positive integer n there exists $\delta>0$ such that the following holds.*

Let X be a compact n-polyhedron and Y its $(n-1)$-subpolyhedron, Q a PL m-manifold with boundary, $m-n\geq 3$, and $f:(X,Y)\times[-1,1]\to(Q,\partial Q)\times\mathbb{R}$ a PL embedding such that $\Pi_2\circ f$ is δ-close to π. Then for each $\gamma>0$ there is a PL ambient isotopy H_t with support in $Q\times[-\varepsilon,\varepsilon]$ such that

H_t moves points less than ε,

$\Pi_1\circ H_t$ moves points less than γ,

H_t takes f onto a PL embedding g such that $g^{-1}(Q\times J)=X\times(J\cap[-1,1])$ for each $J=(-\infty,0]$, $\{0\}$, $[0,+\infty)$.

Moreover, if $f^{-1}(\partial Q\times J)=Y\times J$ for each J as above, then H_t can be chosen to fix $\partial Q\times\mathbb{R}$ [Ed75, Lemma 4.1].

Proof of (k,l). Choose any vertex $v\in\mathbb{R}^m-C$. Let $D=\mathrm{st}(v,T')$ be its dual cell, and denote $E=\mathrm{Cl}(\partial D-\partial C)$. Notice that the pair $(E,\partial E)$ is bi-collared in $(\mathrm{Cl}(\mathbb{R}^m-C),\partial C)$. By Edwards' Slicing Lemma, for any $\varepsilon_1>0$ the number $\delta<\varepsilon_1$ can be chosen so that f is PL ε_1-ambient isotopic, keeping C fixed, to a PL embedding $h_1:X\to\mathbb{R}^m$ such that $h_1^{-1}(E)=s^{-1}(E)$. It follows that, in addition, $h_1^{-1}(D)=s^{-1}(D)$. Note that h_1 is $(\varepsilon_1+\delta)$-close to s and $\varepsilon_1+\delta<2\varepsilon_1$.

Consider the triangulation of D defined from T by a pseudo-radial projection [RS72] $\partial D \to \partial\,\mathrm{st}(v,T)$ (note that in general $\partial\,\mathrm{st}(v,T) \ne \mathrm{lk}(v,T)$). Apply (k-1,l') in ∂D equipped with the above triangulation, where l' is the number of dual cells in $C \cap \partial D$. Using collaring, we obtain that for each $\varepsilon_2 > 0$ we can choose $\varepsilon_1 < \varepsilon_2$ so that h_1 (which is $2\varepsilon_1$-close to s) is PL ε_2-ambient isotopic, keeping C fixed, to a PL embedding $h_2 : X \to \mathbb{R}^m$ such that $h_2^{-1}(\sigma) = s^{-1}(\sigma)$ for each dual cell σ of $C \cup D$. Note that h_2 is $(\varepsilon_2 + 2\varepsilon_1)$-close to s and $\varepsilon_2 + 2\varepsilon_1 < 3\varepsilon_2$.

By (k,l-1), the number $\varepsilon_2 < \varepsilon/3$ can be chosen so that h_2 (which is $3\varepsilon_2$-close to s) is PL $\varepsilon/3$-ambient isotopic, keeping $C \cup D$ fixed, to a PL embedding $h : X \to \mathbb{R}^m$ such that $h^{-1}(\sigma) = s^{-1}(\sigma)$ for each dual cell σ of T. Thus f is ε-ambient isotopic to h, keeping C fixed, because $\varepsilon/3 + 2\varepsilon_2 < \varepsilon$. $\square$

Although the statement of the Fibering Lemma 7.1 does not involve ε-control as in the Slicing Lemma 7.2, this control is used in the proof.

We conjecture that the Fibering Lemma 7.1 is true even for TOP locally flat embeddings. Analogously to the Fibering Lemma 7.1 it is proved that if a PL manifold N is a D^p-bundle over another manifold, then every embedding $N \to \mathbb{R}^m$ in codimension at least 3 is isotopic to a fiberwise embedding (which is defined analogously).

(12) *An idea how to define $\sigma(f)$.* Let N be a compact n-manifold. Fix an n-ball B in the interior of N. Take a proper PL embedding $f : N \to D^m$ such that $f_{N - \mathrm{Int}\, B^n}$ is a smooth embedding. If we define a smooth regular neighborhood B^m of $f(B^n)$ relative to $f(\partial B^n)$, we can define $\sigma(f)$ be the isotopy class of the abbreviation $\partial B^n \to \partial B^m$ of f (and try to prove that this is independent on the choice of B^m).

(13) *Sketch of a new proof that $KT^{3l}_{0,2l-1,PL} \cong \pi_{2l-1}(S^l) \oplus \mathbb{Z}_{(l)}$ for $l \ge 2$.* Analogously to the proof of Realization Theorem 2.3.b considering the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_{(l)} \arrow[r, "\omega"'] \arrow[d, "i"] & KT^{3l}_{0,2l-1} \arrow[r, "\lambda"'] & \pi_{2l-1}(S^l) \arrow[r] & 0 \\
& {} \arrow[r, "i"'] & \pi_{2l-1}(S^l) \oplus \mathbb{Z}_{(l)} \arrow[u, "\tau\oplus\omega"] \arrow[r, "j"'] & {} \arrow[u, "j"] &
\end{tikzcd}$$

Here $i(x) := [(0,x)]$, $j[(a,b)] := [a]$ (clearly, j is well-defined by this formula) and λ is the linking coefficient.

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Department of Differential Geometry, Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia 119992, and Independent University of Moscow, B. Vlasyevskiy, 11, 119002, Moscow, Russia. e-mail: skopenko@mccme.ru